# Fisher's 4 unit maths specimen papers

## PAPER 1

### Question 1

**(a)** Evaluate: (i) $\int_0^1 \frac{dx}{(x+1)(x+3)}$; (ii) $\int_0^1 \sqrt{(4-x^2)}\,dx$; (iii) $\int_{-1}^2 x\sqrt{2-x}\,dx$

**(b)** Find $\int x^2 e^{-x}\,dx$

**(c)** In $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$, show that $I_n = \frac{n-1}{n} I_{n-2}$. Hence evaluate $\int_0^{\frac{\pi}{2}} \sin^5 x\,dx$.

### Question 2

**(a)** Reduce the polynomial $P(x) = x^4 - 2x^2 - 15$ into irreducible factors over: (i) the rational field $\mathbb{Q}$; (ii) the real field $\mathbb{R}$; (iii) the complex field $\mathbb{C}$.

**(b)** Divide the polynomial $(x^3 + 5ix^2 - 7ix - 3)$ by $(x - 2i)$ using long division.

**(c)** Show that $2 - \sqrt{3}$ is a zero of the polynomial $a(x) = x^3 - 15x + 4$. Hence reduce $a(x)$ to irreducible factors over the real field.

**(d)** Given that the polynomial $P(x) = x^4 + x^2 + 6x + 4$ has a rational zero of multiplicity 2, find all the zeros of $P(x)$ over the complex field.

**(e)** If $\alpha, \beta, \gamma$ are the roots of the equation $x^3 + qx + r = 0$ where $r \neq 0$, obtain as functions of $q$ and $r$, in their simplest forms, the coefficients of the cubic equation whose roots are $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$.

### Question 3

**(a)** (i) Define the modulus $|z|$ of a complex number $z$.
(ii) Given two complex numbers $z_1, z_2$, prove that $|z_1 z_2| = |z_1||z_2|$.

**(b)** Given $w = \frac{2-3i}{1-i}$, determine:
(i) $|w|$ (the modulus of $w$); (ii) $\overline{w}$ (the conjugate of $w$); (iii) $w + \overline{w}$.

**(c)** Describe, in geometric terms, the locus (in the Argand plane) represented by $2|z| = z + \overline{z} + 4$.

### Question 4

**(a)** Determine the real values of $k$ for which the equation $\frac{x^2}{19-k} + \frac{y^2}{7-k} = 1$ defines respectively an ellipse and a hyperbola. Sketch the curve corresponding to the value $k = 3$. Describe how the shape of this curve changes as $k$ increases from 3 towards 7. What is the limiting position of the curve as 7 is approached?

**(b)** $P$ is a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with centre $O$. A line is drawn through $O$, parallel to the tangent to the ellipse at $P$, meets the ellipse at $Q$ and $R$. Prove that the area of triangle $PQR$ is independent of the position of $P$.

### Question 5

**(a)** Sketch the curve $y^2 = x^2(x-2)(x-3)$.

**(b)** In the Cartesian plane sketch the curve $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ and prove that the lines $y = \pm 1$ are asymptotes. Also, if $k$ is a positive constant, find the area in the positive quadrant enclosed by the above curve and the three lines $y = 1, x = 0, x = k$, and prove that this area is always less than $\ln 2$, however large $k$ may be.

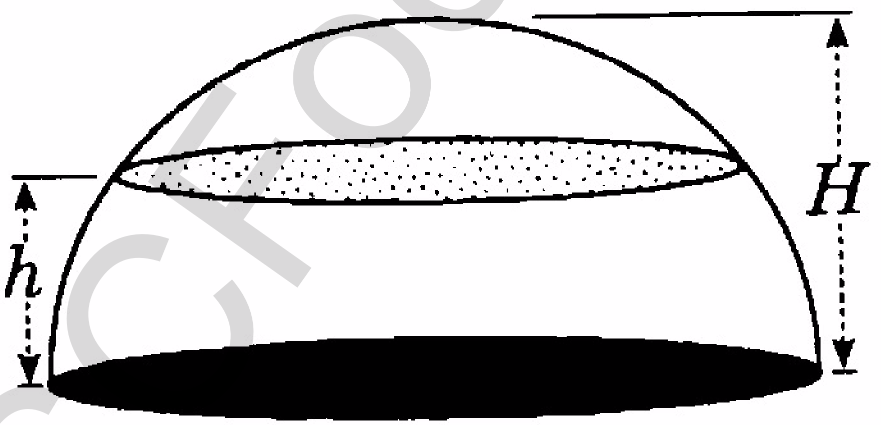
### Question 6

**(a)** The area bounded by the curve $y = \frac{1}{x+1}$, the $x$-axis, the line $x = 2$, and the line $x = 8$, is rotated about the $y$-axis. Find the volume of the solid generated using the method of cylindrical shells.

**(b)** (i) Using substitution $x = a\sin\theta$, or otherwise, verify that $\int_0^a \sqrt{a^2 - x^2}\, dx = \frac{1}{4}\pi a^2$

(ii) Deduce that the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\pi ab$.

(iii)



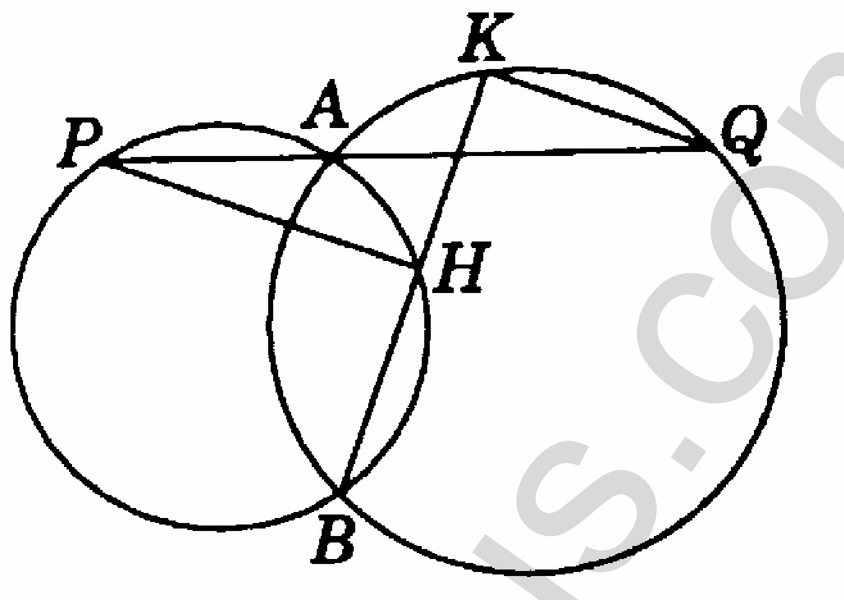

The diagram shows a mound of height $H$. At height $h$ above the horizontal base, the horizontal cross-section of the mound is elliptical in shape, with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \lambda^2$, where $\lambda = 1 - \frac{h^2}{H^2}$ and $x, y$ are appropriate coordinates in the plane of the cross-section. Show that the volume of the mound is $\frac{8\pi abH}{15}$.

### Question 7

**(a)** Six letters are chosen from the letters of the word AUSTRALIA. These six letters are then placed alongside one another to form a six-letter arrangement. Find the number of distinct six-letter arrangements which are possible, considering all the choices.

**(b)** Solve for $x$ the following inequation: $\frac{x^2-5x}{4-x} \leq -3$. Show the solutions on a number line.

**(c)**




In the figure, $PAQ$ and $BHK$ are straight lines. Prove that $PH$ is parallel to $KQ$.

**(d)** Two circles, centres $B$ and $C$, touch externally at $A$. $PQ$ is a direct common tangent touching the circles at $P$ and $Q$ respectively.

(i) Draw a neat diagram depicting the given information.
(ii) Prove that the circle with $BC$ as its diameter touches the line $PQ$.

**Question 8**

**(a)** An aeroplane flies horizontally due east at a constant speed of 240 km/h. From a point $P$ on the ground the bearing of the plane at one instant is 311°T, and 3 minutes later the bearing of the plane is 073°T, while its elevation then is 21°. If $h$ metres is the altitude of the plane, show that $h = 12000 \sin 41° \tan 21° \operatorname{cosec} 58°$, and calculate $h$ correct to the nearest metre.

**(b)** The magnitude and direction of the acceleration due to gravity at a point uotside the Earth at a distance $x$ from the Earth's centre is equal to $-\frac{k}{x^2}$, where $k$ is a constant.

(i) Neglecting atmospheric resistance, prove that if an object is projected upwards from the Earth's surface with speed $u$, its speed $v$ in any position is given by $v^2 = u^2 - 2gR^2\left(\frac{1}{R} - \frac{1}{x}\right)$ where $R$ is the Earth's radius and $g$ is the magnitude of the acceleration due to gravity at the Earth's surface.

(ii) Show that the greatest height, $H$, above the Earth's surface reached by the particle is given by $H = \frac{u^2 R}{2gR - u^2}$.

(iii) Hence, if the radius of the Earth is approximately 6400 km, and the acceleration due to gravity at the Earth's surface is $9 \cdot 8$ m/s$^2$, find the speed required by the particle to escape the Earth's gravitational influence.

## PAPER 2

### Question 1

Sketch the following curves on separate axes, showing all intercepts and turning points:

**(a)** $y = \cos^2 x$ **(b)** $y = \ln|\cos x|$ **(c)** $y = x^3 - 4x$

**(d)** $y = (x^3 - 4x)^2$ **(e)** $y^2 = x^3 - 4x$ **(f)** $y = |x|^3 - 4|x|$.

### Question 2

**(a)** Given the complex number $z = 7 - 3i$, find:

(i) $|z|$ (ii) $\overline{z}$ (iii) $|z - \overline{z}|$ (iv) $\arg(z - \overline{z})$.

**(b)** Express $z = \frac{\sqrt{2}}{1-i}$ in modulus-argument form, and hence express $z^5$ in the form $x + iy$.

**(c)** $K, L, M, N$ are the vertices of a square, in anticlockwise order. Given that $K$ and $M$ represent the numbers $2 + i$ and $2 + 3i$, find the coordinates of:

(i) $L$ and $N$
(ii) $M$, if the square is rotated clockwise through an angle of $90°$ about the origin.

**(d)** In the Argand plane, sketch the following:

(i) $|z - 5 + 3i| = 5$
(ii) $|z^2 - \overline{z}^{\,2}| \geq 4$
(iii) $\arg \frac{z+1}{z-1} = 0$

### Question 3

**(a)** Derive the equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(x_1, y_1)$ and hence deduce that the equation of the chord of contact to this hyperbola from an external point $E(x_0, y_0)$ is $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$. If the chord of contact passes through a focus, show that $E$ lies on a directrix, and determine whether the chord subtents a right angle at $E$.

**(b)** Let $C_1 \equiv 3x^2 + y^2 - 1, C_2 \equiv 2x^2 + 5y^2 - 1$, and let $\lambda$ be a real number.

(i) Show that $C_1 + \lambda C_2 = 0$ is the equation of a curve passing through the points of intersection of the ellipses $C_1 = 0$ and $C_2 = 0$.

(ii) Determine the values of $\lambda$ for which $C_1 + \lambda C_2 = 0$ is the equation of an ellipse.

### Question 4

Leaving your answer in exact form, evaluate:

**(a)** $\int_0^1 \frac{dx}{x^2+8x+4}$ **(b)** $\int_1^e \ln x^3 \ dx$ **(c)** $\int_0^{\frac{\pi}{2}} \frac{dx}{\sin x+1}$

**(d)** $\int_0^2 \sqrt{\frac{5-x}{5+x}} \ dx$ **(e)** $\int_3^5 \frac{3 \ dx}{2x^3+x^2-x}$.

## Question 5

**(a)** Sketch the region, $R$, which is completely bounded by the curves $y = \sin 2x$ and $y = \frac{1}{2}$ in the domain $0 \leq x \leq \frac{\pi}{2}$. Find the volume generated when $R$ is rotated about the:

(i) $x$-axis (ii) $y$-axis (iii) line $y = \frac{1}{2}$.

**(b)** A solid shape has a triangular base with sides $17$ cm, $17$ cm and $16$ cm. Each cross-section (perpendicular to the axis of symmetry of the base) is in the shape of a parabola, with its latus rectum lying in the base. Find the volume of the solid.

## Question 6

**(a)** The polynomial function $P(x) = x^4 - 4x^3 - 3x^2 + 50x - 52$ has a zero at $x = 3 - 2i$. Factorise $P(x)$ over the field of:

(i) rationals
(ii) reals
(iii) complex numbers

**(b)** The diameter $AB$ of a circle is produced to $E$. $EC$ is a tangent touching the circle at $C$, and the perpendicular to $AE$ at $E$ meets $AC$ produced at $D$. Show that $\triangle CDE$ is isosceles.

**(c)** Given that $p, q$, and $r$ are positive, prove:

(i) $p^2q + pq^2 + q^2r + qr^2 + r^2p + rp^2 \geq 6pqr$
(ii) $(p+q)(q+r)(r+p) \geq 8pqr$
(iii) $\left(\frac{1}{p} - 1\right)\left(\frac{1}{q} - 1\right)\left(\frac{1}{r} - 1\right) \geq 8$ if also $p + q + r = 1$.

## Question 7

**(a)** A sequence $\{u_n\}$ is such that $u_1 = 3, u_2 = 5$ and $u_{n+2} = 4u_{n+1} - 3u_n$. Prove by mathematical induction that $u_n = 3^{n-1} + 2$.

**(b)** If $\alpha, \beta$ and $\gamma$ are angles of a triangle, prove $\cos\alpha + \cos\beta + \cos\gamma = 4\sin\frac{\alpha}{2}\sin\frac{\alpha}{2}\sin\frac{\gamma}{2} + 1$.

**(c)** How many different sums of money can be made up from six \$20 notes, two \$5 notes, four \$2 coins, seven \$1 coins, three 50¢ coins, and one 20¢ coin?

**(d)** Five letters are chosen from the letters of the word AMERICA. These five letters are then placed alongside one another to form a five-letter arrangement. Find the number of distinct five-letter arrangements which are possible, considering all the choices.

**Question 8**

**(a)** (i) A particle of mass $m$ is projected vertically downwards under gravity in a medium whose resistance is equal to the velocity of the particle multiplied by $\frac{mg}{V}$. Show that the velocity tends to the value $V$.

(ii) A particle is projected vertically upwards in the above medium with velocity $U$. Show that it reaches a height $\frac{UV}{g} + \frac{V^2}{g}\log_e\left(\frac{V}{U+V}\right)$.

**(b)** A car takes a banked curve of a racing track at a speed $V$, the lateral gradient angle $\theta$ being designed to reduce the tendency to side-slip to zero for a lower speed $U$. Show that the coefficient of friction necessary to prevent side-slip for the greater speed $V$ must be at least $\frac{(V^2-U^2)\sin\theta\cos\theta}{V^2\sin^2\theta+U^2\cos^2\theta}$.

# PAPER 3

## Question 1

Sketch the following curves on separate axes, showing all intercepts and turning points:

**(a)** $y = \sin x$, hence $y^2 = \sin x$ (in the domain $-2\pi \leq x \leq 2\pi$)

**(b)** $y = 4x - x^3$, hence $y = 4|x| - |x^3|$ (in the domain $-3 \leq x \leq 3$)

**(c)** $y = \ln|\tan x|$, (in the domain $-\frac{3\pi}{2} \leq x \leq \frac{3\pi}{2}$)

**(d)** $y = \frac{x^2-2}{x^2-1}$, (in the domain $-3 \leq x \leq 3$)

## Question 2

**(a)** Evaluate $\int_{-1}^{0} x^3\sqrt{2-x^2}\,dx$

**(b)** Find the following integrals:

(i) $\int \frac{dx}{(x+1)(x+3)^2}$ (ii) $\int \frac{dx}{1+\sin x}$

**(c)** Find the exact value of $\int_0^1 xe^{x^2}\,dx$

**(d)** Given $I_{2n+1} = \int_0^1 x^{2n+1}e^{x^2}\,dx$ where $n$ is a positive integer, show that $I_{2n+1} = \frac{1}{2}e - nI_{2n-1}$. Hence, or otherwise, evaluate $\int_0^1 x^5e^{x^2}\,dx$.

## Question 3

**(a)** Express $-1 + \sqrt{3}i$ in mod-arg form.

**(b)** Given $z = \frac{1+i\sqrt{3}}{1+i}$

(i) express $z$ in the form $a + ib$

(ii) show that $z = \sqrt{2}\big(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\big)$

(iii) find exact values of $\cos\frac{\pi}{12}$ and $\sin\frac{\pi}{12}$

**(c)** On an Argand diagram, the points $P$ and $Q$ represent the numbers $z_1$ and $z_2$ respectively. $OPQ$ is an equilateral triangle. Show that $z_1^2 + z_2^2 = z_1z_2$.

## Question 4

**(a)** For the rectangular hyperbola $xy = 12$, find:

(i) the eccentricity
(ii) the coordinates of the foci
(iii) the equations of the directrices
(iv) the equations of the asymptotes
(v) sketch the hyperbola

**(b)** $P(a\sec\theta, b\tan\theta)$ lies on the rectangular hyperbola $x^2 - y^2 = a^2$. $A$ is the point $(a, 0)$. $M$ is the midpoint of $AP$. Find the equation of the locus of $M$.

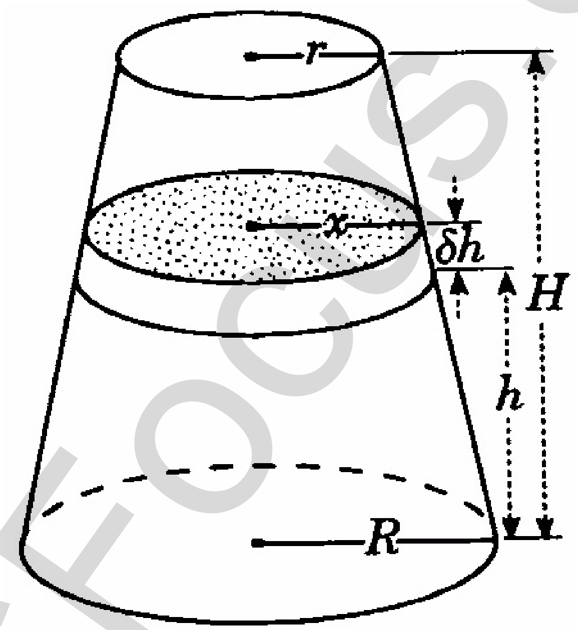
**(c)** $P(ct, \frac{c}{t})$ lies on the rectangular hyperbola $xy = c^2$. The normal at $P$ meets the hyperbola again at $Q$. $M$ is the midpoint of $PQ$. Find the equation of the locus of $M$.

**Question 5**

**(a)** Use the method of cylindrical shells to find the volume of a torus (doughnut) with inner radius 3 cm and outer radius 5 cm.

**(b)** The base of a solid is the segment of the parabola $x^2 = 4y$ cut off by the line $y = 2$. Each cross-section (perpendicular to the axis of the parabola) is a right-angled isosceles triangle with hypotenuse in the base of the solid. Find the volume of the solid.

**(c)**




Calculate the volume of the frustrum of a cone, with radii of the top and bottom circles being $r$ and $R$ respectively, and the height of the frustrum being $H$.

**Question 6**

**(a)** The polynomial function $P(x) = x^4 - 2x^3 + 6x^2 - 2x + 5$ has a zero at $x = 1 + 2i$. factorise $P(x)$ over the field of:

(i) rationals
(ii) reals
(iii) complex numbers

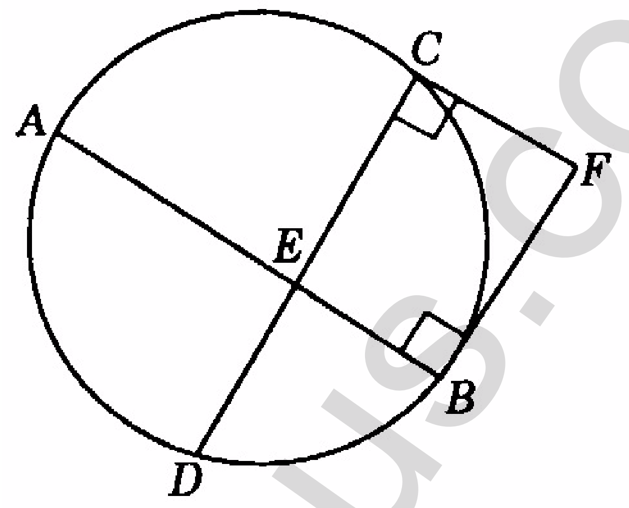
**(b)** The equation $x^4 + 4x^3 - 3x^2 - 4x - 2 = 0$ has roots $\alpha, \beta, \gamma, \delta$. Find the equations with roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$.

**(c)** The equation $2x^3 - 9x^2 + 7 = 0$ has roots $\alpha, \beta, \gamma$. Find the equation with roots $\alpha^3, \beta^3, \gamma^3$.

**(d)** Solve $x^5 + 2x^4 - 2x^3 - 8x^2 - 7x - 2 = 0$ if it has a root of multiplicity 4.

### Question 7

**(a)** In the figure, $AB$ and $CD$ are two chords of the circle. $AB$ and $CD$ intersect at $E$. $F$ is a point such that $A\hat{B}F$ and $D\hat{C}F$ are right angles.




Prove that $FE$ produced is perpendicular to $AD$.

**(b)** Two circles intersect in $A$ and $B$. $C$ and $D$ are points on the respective circles such that $\angle CAB = \angle DAB$. $CB$ and $DB$ are produced to cut the circles again at $E$ and $F$.

(i) Draw a neat diagram depicting the given information.
(ii) Show that $BC.BE = BD.BF$.

**(c)** Show that $(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9$

**(d)** Using calculus, show that $x \geq \ln(1 + x)$ for $x \geq -1$.

### Question 8

**(a)** A particle of mass 10 kg is found to experience a resistive force, in newtons, of one-ninth of the square of its velocity in metres per second when it moves through the air. The particle is projected vertically upwards from a point $O$ with a velocity of $30\sqrt{3}$ m/s and the point $A$, vertically above $O$, is the highest point reached by the particle before it starts to fall to the ground again. Assuming the value of $g$ is 10 m/s$^2$:

(i) find the time the particle takes to reach $A$ from $O$.
(ii) find the height $OA$.

**(b)** The railway line around a circular arc of radius 800 m is banked by raising the outer rail to a level above the inner rail. When the train travels at 10 m/s, the lateral thrust on the inner rail is the same as the lateral thrust on the outer rail at a speed of 20 m/s. Calculate the angle of banking and the speed of the train when there is no lateral thrust exerted on the rails. (Use $g = 9{\cdot}8$ m/s$^2$.)



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