Arnold & Arnold's
4 unit maths specimen papers 1-6

PAPER 1

1 (a) (i) Sketch the graph of $f(x) = x \ln x$

(ii) Use your graph to sketch the graph of $h(x) = \frac{1}{x \ln x}$

(b) (i) Sketch the graph of the function $g(x) = x \ln x$ for $x \geqslant \frac{1}{e}$ showing clearly the coordinates of its endpoint and its points of intersection with the x-axis and the line $y = x$.

(ii) On the same axes, sketch the graph of the inverse function $g^{-1}(x)$, showing clearly the coordinates of its endpoint and its points of intersection with the y-axis and the line $y = x$.

(iii) Evaluate $\int_1^e x \ln x \ dx$. Hence find the area bounded by the x-axis between $x = 0$ and $x = 1$, the y-axis between $y = 0$ and $y = 1$, and the graphs of $y = g(x)$ and $y = g^{-1}(x)$.

2 (a) Use the substitution $u = e^x$ to find $\int \frac{e^{2x}}{1+e^x} \ dx$

(b) Find $\int \frac{\sin^3 x}{\cos^4 x} \ dx$

(c) Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{4+5\sin x} \ dx$

(d) Evaluate $\int_0^1 \frac{\sin^{-1} x}{\sqrt{1+x}} \ dx$

3 (a) Show that the gradient of the tangent to the hyperbola $\frac{x^2}{9} - \frac{y^2}{7} = 1$ at the extremety in the first quadrant of its latus rectum is equal to the eccentricity of the hyperbola.

(b) $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b > 0$. The tangent and the normal at P cut the y-axis at A and B respectively, and S is a focus of the ellipse.

(i) Show that $A\widehat{S}B = 90°$

(ii) Hence show that A, P, S and B are concyclic and state the location of the centre of the circle through A, P, S and B.

4 (a) (i) Find real numbers a and b such that $(a + ib)^2 = -3 + 4i$

(ii) Hence solve the equation $z^2 - 3z + (3 - i) = 0$

(b) **(i)** Express each of $1+\sqrt{3}i$ and $1-\sqrt{3}i$ in modulus argument form.

(ii) Hence simplify $(1+\sqrt{3}i)^{10}+(1-\sqrt{3}i)^{10}$

(c) **(i)** By considering the points of intersection of the curve $y=x^3-x+2$ and the line $y=mx$, show that there is only one tangent to the curve which passes through the origin.

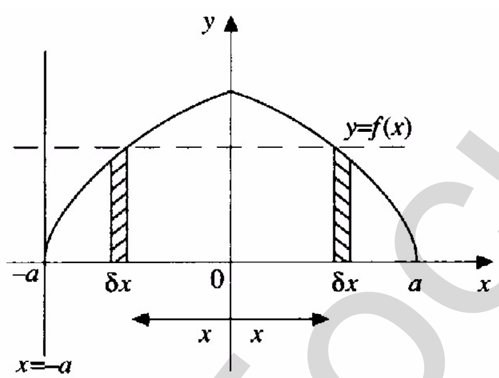
(ii) Find the equation of this tangent and its point of contact with the curve.

5 **(a)** On separate Argand diagrams, shade in the regions containing all points representing complex numbers z such that

(i) $|z| \leqslant 1$ and $0 \leqslant \arg z \leqslant \frac{\pi}{4}$

(ii) $|z| \leqslant 1$ or $0 \leqslant \arg z \leqslant \frac{\pi}{4}$

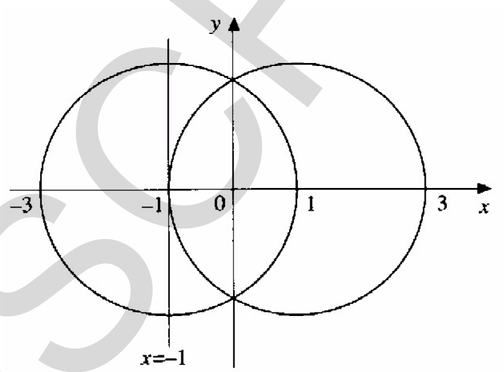
(b) **(i)**



$f(x)$ is an even function. The area bounded by $y=f(x)$ and the x-axis is rotated about $x=-a$. The strips of width δx form cylindrical shells of the same height.

Show that the volume of the solid is given by $V=4\pi a\int_0^a f(x)\,dx$.

(ii)



The centres of two circles, each of radius 2 cm, are 2 cm apart. The region common to the two circles is rotated about one of the tangents to this region which is perpendicular to the line joining the centres.

Show that the volume of the solid formed is given by $V=8\pi\int_0^1\sqrt{4-(x+1)^2}\,dx$, and hence find this volume.

6 The ends of a light string are fixed to two points A and B in the same vertical plane with A above B, and the string passes through a small smooth ring of mass m. The ring is fastened to the string at a point P and when the string is taut, $A\widehat{P}B=90^\circ, B\widehat{A}P=\theta$ and the distance of P from AB is r. The ring revolves in a horizontal circle with constant angular velocity ω and with thre string taut.

(i) Find the tensors T_1 and T_2 in the parts AP and PB of the string. Hence, given that $AB = 5\ell$ and $AP = 4\ell$, show that $16\ell\omega^2 > 5g$.

(ii) If the ring is free to move on the string, instead of being as before, revolving in a horizontal circle with constant angular velocity Ω, then Ω satisfies the equation $12\ell\Omega^2 = 35g$.

7 (a) (i) Show that the remainder when the polynomial $P(x)$ is divided by $(x-a)^2$ is $P'(a)x + P(a) - aP'(a)$.

(ii) Find thye value of k for which $x-1$ is a factor of the polynomial $P(x) = x^{11} - 3x^6 + kx^4 + x^2$. For this value of k, find the remainder on dividing $P(x)$ by $(x-1)^2$.

(b) Owing to the tides, the height of water in an esuary may be assumed to rise and fall with time in simple harmonic motion. At a certain place there is a danger of flooding when the height of water is above 1.25 m. One day the high tide had a height of 1.5 m at 1.00 am and the following low tide had a height of 0.5 m at 7.30 am. Assuming that the following high tide also had a height of 1.5 m, find the times that day when there was a danger of flooding.

8 (a) If $u_1 = 5, u_2 = 13$ and $u_n = 5u_{n-1} - 6u_{n-2}$ for $n \geqslant 3$, show that $u_n = 2^n + 3^n$ for $n \geqslant 1$.

(b) (i) If $a > 0, b > 0$, show that $a + b \geqslant 2\sqrt{ab}$.

(ii) Hence show that

- If $a > 0, b > 0$ and $c > 0$, then $(a+b)(b+c)(c+a) \geqslant 8abc$
- If $a > 0, b > 0, c > 0$ and $d > 0$, then $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geqslant 4$.

PAPER 2

1 (a) (i) Find the domain and the range of the function $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 1$. Sketch the graph of the function.

(ii) $P(x_1, y_1)$ lies on the curve $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 1$. The tangent at P meets the x-axis and the y-axis at Q and R respectively. Show that $OQ + OR$ is independent of the position of P.

(b) (i) Find the gradient of the tangent to the curve $y = e^x$ which passes through the origin.

(ii) Hence find the values of the real number k for which the equation $e^x = kx$ has exactly two real solutions.

2 (a) Express $3 + 2x - x^2$ in the form $b^2 - (x - a)^2$. Hence evaluate $\int_1^3 \sqrt{3 + 2x - x^2}dx$

(b) Find $\int x \sec^2 x \; dx$

(c) Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x+\sin x} \; dx$. Hence use the substitution $u = \frac{\pi}{2} - x$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{x}{1+\cos x+\sin x} \; dx$.

3 (a) Show that the equation $\frac{x^2}{29-k} + \frac{y^2}{4-k} = 1$, where k is a real number, represents

(i) an ellipse if $h < 4$

(ii) a hyperbola if $4 < k < 29$

Show that the foci of each ellipse in **(i)** and each hyperbola in **(ii)** are independent of the value of k.

(b) $P\big(ct, \frac{c}{t}\big)$, where $t \neq 1, t \neq -1$, lies on the rectangular hyperbola $xy = c^2$. The tangent at P meets the x-axis and the y-axis at Q and R respectively. The normal at P meets the lines $y = x$ and $y = -x$ at S and T respectively. Show that $QSRT$ is a rhombus.

4 (a) (i) Express the complex numbers $z_1 = 2i$ and $z_2 = -1 + \sqrt{3}i$ in modulus argument form. On an Argand diagram, plot the points P and Q which represent z_1 and z_2 respectively.

(ii) On the same diagram, construct the vectors which represent the complex numbers $z_1 + z_2$ and $z_1 - z_2$ respectively. Deduce the exact values of $\arg(z_1 + z_2)$ and $\arg(z_1 - z_2)$.

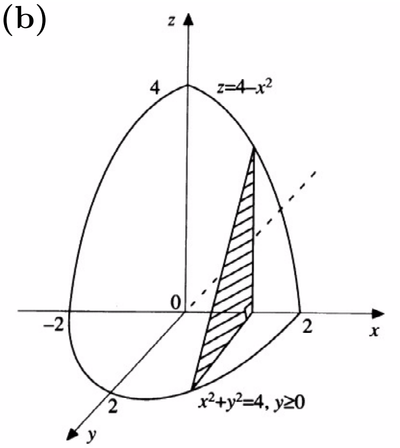
(b) If $\arg(z-2) - \arg(z+2) = \frac{\pi}{4}$, show that the locus of the point P representing the complex number z is an arcf of a circle, and find the centre and the radius of this circle.

(c) The polynomial $P(x) = x^3 + ax^2 - x - 2$, where a is a constant, has three real zeros, one of which is twice another. Find the value of a and factorise $P(x)$ over the real numbers.

5 (a) (i) If $z = \cos\theta + i\sin\theta$, show that $z^n + z^{-n} = 2\cos n\theta$

(ii) Hence show that $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$

(b)



The solid shown has a semicircular base of radius 2 units. Vertical cross-sections perpendicular to the diameter are right-angled triangles whose height is bounded by the parabola $z = 4 - x^2$.

By slicing at right angles to the x-axis, show that the volume of the solid is given by $V = \int_0^2 (4 - x^2)^{\frac{3}{2}}\ dx$, and hence calculate this volume.

6 Two particles are connected by a light inextensible string which passes through a small hole with smooth edges in a smooth horizontal table. One particle of mass m travels on the table with constant angular velocity ω. Another particle of mass M travels in a circle with constant angular velocity Ω on a smooth horizontal floo, distance x below the table. The lengths of string on the table and below the table are ℓ and L respectively and the length L makes an angle θ with the vertical.

(a) (i) If the floor exerts a force N on the lower particle, show that $N = M(g - x\Omega^2)$. Find the maximum possible value of Ω for the motion to continue as described. What happens if Ω exceeds this value?

(ii) By considering the tension force in the string, show that $\frac{L}{\ell} = \frac{m}{M}\left(\frac{\omega}{\Omega}\right)^2$. If the lower particle exerts zero force on the floor, show that the tension force in the string is given by $T = \frac{MgL}{x}$.

(b) The table is 80 cm high and the string is 1.5 m long, while the masses on the table and on the floor are 0.4 kg and 0.2 kg respectively. The particles are observed to have the same angular velocity. If the lower particle exerts zero force on the floor, find

(i) the tension in the string

(ii) the speed of the particle on the table if the string were to break.

7 (a) Show that the remainder when the polynomial $P(x)$ is divided by x^2-a^2 is $\frac{1}{2a}\{P(a)-P(-a)\}x+\frac{1}{2}\{P(a)+P(-a)\}$. Find the remainder when $P(x)=x^n-a^n$ is divided by x^2-a^2 in each of the cases **(i)** n even **(ii)** n odd.

(b) A particle P is projected from a point O and inclined at an angle of $45°$ above the horizontal. The particle describes a parabola under gravity. Coordinate axes are taken horizontally and vertically through O. The particle just clears the tops of two vertical poles a distance 40 m apart and each 15 m above the point of projection. Find the horizontal range of the projectile.

8 (a) Show that 7^n+15^n is divisible by 11 for all odd $n \geqslant 1$.

(b) (i) If $a>0$ and $b>0$, show that $a+b \geqslant 2\sqrt{ab}$

(ii) If also $a+b=1$, show that $\frac{1}{a}+\frac{1}{b} \geqslant 4$ and $\frac{1}{a^2}+\frac{1}{b^2} \geqslant 8$.

PAPER 3

1 (i) Sketch the graph of $y = \frac{3x}{x^2-1}$

(ii) Solve the equation $\frac{3x}{x^2-1} = 2$. Use your graph to solve the inequality $\frac{3x}{x^2-1} > 2$.

(iii) Find the gradient of the curve $y = \frac{3x}{x^2-1}$ at the origin. Use your graph to find the values of the negative real number k for which the equation $\frac{3x}{x^2-1} = kx$ has exactly one real solution.

(iv) Find the area of the region bounded by the curve $y = \frac{3x}{x^2-1}$, the x-axis and the line $x = \frac{1}{2}$.

2 (a) Find $\int \frac{x+3}{(x+1)(x^2+1)}\, dx$

(b) Use the substitution $x = 4\sin^2\theta$ to evaluate $\int_0^2 \sqrt{x(4-x)}\, dx$

(c) (i) If $I_n = \int_0^1 x^n e^x\, dx$ for $n \geqslant 0$ show that $I_n = e - nI_{n-1}$ for $n \geqslant 1$

(ii) Find the value of I_4 and hence evaluate $\int_0^{\frac{1}{2}} x^4 e^{2x}\, dx$

3 (a) $P\big(ct, \frac{c}{t}\big)$ lies on the rectangular hyperbola $xy = c^2$.

(i) Show that the normal at P cuts the hyperbola again at the point Q with coordinates $\big(-\frac{c}{t^3}, -ct\big)$. Hence find the coordinates of the point R where the normal at Q cuts the hyperbola again.

(ii) The normal at P meets the x-axis at A and the tangent at P meets the y-axis at B. M is the midpoint of AB. Find the equation of the locus of M as P moves on the hyperbola.

4 (a) The complex number z and its conjecture $\overline{z}$ satisfy the equation $z\overline{z}+2iz = 12+6i$. Find the possible values of z.

(b) Find in modulus argument form the five roots of $z^5 = -1$.

(i) Show that when these five roots are plotted on an Argand diagram they form the vertices of a regular pentagon of area $\frac{5}{2}\sin\frac{2\pi}{5}$.

(ii) By combining appropriate pairs of roots show that $z^4 - z^3 + z^2 - z + 1 = \big(z^2 - 2z\cos\frac{\pi}{5} + 1\big)\big(z^2 - 2z\cos\frac{3\pi}{5} + 1\big)$. Deduce that $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$ are the roots of the equation $4x^2 + 2x - 1 = 0$, and hence find their values.

5 (a) Find integers m and n such that $(x+1)^2$ is a factor of the polynomial $P(x) = x^5 + 2x^2 + mx + n$

(b) The equation $x^3 - px - q = 0$ has roots α, β and γ

(i) Show that $\alpha^2 + \beta^2 + \gamma^2 = 2p$

(ii) Express $\alpha^3+\beta^3+\gamma^3$ and $\alpha^5+\beta^5+\gamma^5$ in terms of p and q

(c) The base of a patricular solid is the ellipse $\frac{x^2}{16}+\frac{y^2}{4}=1$. Find the volume of the solid if every cross-section perpendicular to the major axis of the ellipse is an eqiulateral triangle with one side in the base of the solid.

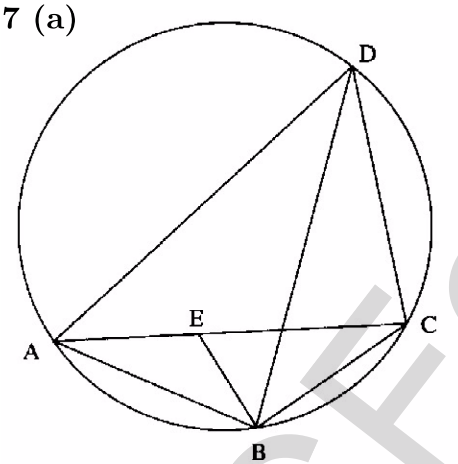
6 (a) A railway line has been constructed around a circular curve of radius 500 m. The distance between the rails is $1{\cdot}5$ m and the outside rail is $0{\cdot}1$ m above the inside rail. Find the speed that eliminates a sideways force on the wheels for a train on this curve. (Take $g=9{\cdot}8$ m.s^{-2}.)

(b) A particle of mass m is set in motion with speed u. Subsequently the only force acting upon the particle directly opposes its motion and is of magnitude $mk(1+v^2)$ where k is a constant and v is its speed at time t.

(i) Show that the particle is brought to rest after time $\frac{1}{k}\tan^{-1}u$

(ii) Find an expression for the distance travelled by the particle in this time.

7 (a)



In the figure, $ABCD$ is a cyclic quadrilateral. E is the point on AC such that $A\widehat{B}E = D\widehat{B}C$.

(i) Show that $\triangle ABC|||\triangle DBC$ and $\triangle ABD|||\triangle EBC$

(ii) Hence show that $AB.DC+AD.BC=AC.DB$

(b) ABC is an equilateral triangle inscribed in a circle. P is a point on the minor arc AB of the circle. Show that $PC=PA+PB$.

8 (a) A tripod is made of three rods OA, OB and OC, each 12 cm long and hinged at O. The ends A, B and C rest on a horizontal table such that each of the angles AOB, BOC and COA is $60°$. Calculate

(i) the height of O above the table

(ii) the tangent of the angle that OA makes with the table.

(b) A fair die is thrown six times. Find the probabilities that the six scores obtained will

(i) be 1,2,3,4,5,6 in some order

(ii) have a product which is an even number

(iii) consist of exactly two 6's and four odd numbers

(iv) be such that a 6 occurs only on the last throw and exactly three of the first five throws result in odd numbers.

PAPER 4

1 (a) (i) If $P(x) = x^3 - 6x^2 + 9x + c$ for some real number c, find the values of x for which $P'(x) = 0$. Hence find the values of c for which the equation $P(x) = 0$ has a repeated root.

(ii) Sketch the graphs of $y = P(x)$ for these values of c. Hence find the set of values of c for which the equation $P(x) = 0$ has only one real root.

(b) (i) Find the domain and the range of the function $f(x) = \cos^{-1}(e^x)$.

bf(ii) Sketch the graph of $y = \cos^{-1}(e^x)$.

2 (a) Express x^2+2x+5 in the form $(x+a)^2+b^2$. Hence evaluate $\int_{-1}^{1} \frac{1}{x^2+2x+5}\, dx$

(b) Find $\int \frac{\ln x}{\sqrt{x}}\, dx$

(c) (i) Show that $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{x(\pi-2x)}\, dx = \frac{2}{\pi}\ln 2$

(ii) Hence use the substitution $u = \frac{\pi}{2} - x$ to evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x}{x(\pi-2x)}\, dx$

3 (a) Show that the ellipse $4x^2 + 9y^2 = 36$ and the hyperbola $4x^2 - y^2 = 4$ meet at right angles. Find the equation of the circle through the points of intersection of these two curves.

(b) $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a > b > 0$. The tangent at P passes through a focus of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Show that it is parallel to one of the lines $y = x$ and $y = -x$ and that its point of contact with the hyperbola lies on a directrix of the ellipse.

4 (a) Express $z_1 = \frac{7+4i}{3-2i}$ in the form $a + ib$, where a and b are real. On an Argand diagram, sketch the locus of the point representing the complex number z such that $|z - z_1| = \sqrt{5}$. Find the greatest value of $|z|$ subject to this condition.

(b) The complex number $z = x + iy$, where x and y are real, is such that $|z - i| = \Im(z)$. Show that the locus of the point representing z has equation $y = \frac{1}{2}(x^1 + 1)$. Find the gradients of the tangents to this curve which pass through the origin, and hence find the set of possible values of $\arg z$

(c) Given that $a + b + c = -3, a^2 + b^2 + c^2 = 29$ and $abc = -6$, form the monic cubic equation whose roots are a, b and c. Hence find the values of a, b and c.

5 (a) Expand the complex number $z = (1 + ic)^6$ in powers of c and hence find the five real values of c for which z is real.

(b) Two of the zeros of the polynomial $P(x) = x^4 + bx^3 + cx^2 + dx + e$, where b, c, d and e are real, are $2 + i$ and $1 - 3i$. Find the other two zeros and hence find the values of b and e.

(c) On the same axes, sketch the graphs of the functions $y = \frac{1}{2}(e^x + e^{-x})$ and $y = \frac{1}{2}(e^x - e^{-x})$. The region between the two curves bounded by the y-axis and the line $x = 1$ is rotated about the y-axis. Use cylindrical shells to show that the volume of the solid generated is given by $V = 2\pi \int_0^1 xe^{-x}\ dx$, and hence calculate this volume.

6 Air resistance to the motion of a particle of mass m has magnitude mkv^2, where v is the speed of the particle and k is a constant.

(i) The particle is projected vertically upward under gravity with initial speed v_0. Write down the equation of motion of the particle during its upward motion and hence show that the greatest height reached is $\frac{1}{2k} \ln\left(\frac{g+kv_0^2}{g}\right)$.

(ii) The particle falls from its greatest height. Write down the equation of motion of the particle during its downward motion and hence find its terminal velocity. If the particle returns to its point of projection with speed v_1, show that $(g + kv_0^2)(g - kv_1^2) = g^2$.

7 (a) (i) Show that $\operatorname{cosec} 2\theta + \cot 2\theta = \cot\theta$ for all real θ.

(ii) Hence find in surd form the values of $\cot\frac{\pi}{8}$ and $\cot\frac{\pi}{12}$, and show that $\operatorname{cosec}\frac{2\pi}{15} + \operatorname{cosec}\frac{4\pi}{15} + \operatorname{cosec}\frac{8\pi}{15} + \operatorname{cosec}\frac{16\pi}{15} = 0$.

(b) The vertices of a quadrilateral $ABCD$ lie on a circle of radius r. The angles subtended at the centre of the circle by the sides AB, BC, CD and DA respectively are in arithmetic sequence with first term α and common difference β.

(i) Show that $2\alpha + 3\beta = \pi$ and interpret this result geometrically.

(ii) Show that the area of the quadrilateral is $2r^2\cos\beta\cos\left(\frac{\beta}{2}\right)$.

8 (a) If $u_1 = 1$ and $u_n = \sqrt{3u_{n-1}}$ for $n \geqslant 2$, show that

(i) $u_n < 3$ for $n \geqslant 1$

(ii) $u_{n+1} > u_n$ for $n \geqslant 1$.

(b) Show that $a^2 + b^2 + c^2 \geqslant ab + bc + ca$. Hence show that $a^4 + b^4 + c^4 \geqslant a^2b^2 + b^2c^2 + c^2a^2 \geqslant abc(a + b + c)$.

PAPER 5

1 (a) Sketch the graph of $y = 3x^4 - 4x^3 - 12x^2$, showing clearly the coordinates of the turning points. Use your graph to find the sets of values of k for which the equation $3x^4 - 4x^3 - 12x^2 - k = 0$ has

(i) no real roots

(ii) just two distinct real roots

(iii) four distinct real roots

(b) (i) Find the value of k for which 1 is a root of the equation $3x^4 - 4x^3 - 12x^2 - k = 0$. For this value of k, find the two consecutive integers between which the other real root α lies.

(ii) By considering the product of the four roots, express the modulus of the non-real complex roots in terms of α and so determine an interval in which the modulus must lie. By considering the sum of the four roots, express the real part of the non-real complex roots in terms of α and so determine an interval in which the real part must lie.

2 (a) Find $\int \frac{x^2}{(x-1)(x-2)}\,dx$

(b) Find **(i)** $\int \cos^3 x\,dx$ **(ii)** $\int \frac{\sin^{-1} x}{1-x^2}\,dx$

(c) Show that $\int_0^1 \frac{\sqrt{x}}{1+x}\,dx = 2 - \frac{\pi}{2}$, and hence evaluate $\int_0^1 \frac{1}{\sqrt{x}} \ln(1+x)\,dx$

3 (a) Show that the chord of contact of the tangents from the point $P_0(x_0, y_0)$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has equation $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$.

(b) Write down the equation of the chord of contact of the tangents from the point $(4, -1)$ to the ellipse $x^2 + 2y^2 = 6$. Hence find the coordinates of the points of contact and the equations of these tangents.

4 (a) If $z_1 = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ and $z_2 = 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$, find the modulus and argument of each of the following:

(i) z_1^3 **(ii)** $\frac{1}{z_2}$ **(iii)** $\frac{z_1^3}{z_2}$

(b) If z is any complex number such that $|z| = 1$, show using an Argand diagram, or otherwise, that

(i) $1 \leqslant |z+2| \leqslant 3$ **(ii)** $-\frac{\pi}{6} \leqslant \arg(z+2) \leqslant \frac{\pi}{6}$.

(c) Find the values of the real numbers p and q if $x^2 + 1$ is a factor of the polynomial $P(x) = x^4 + px^3 + 2x + q$. Hence factorise $P(x)$ over $\mathbb{R}$ and over $\mathbb{C}$.

5 (a) Express $(6+5i)(7+2i)$ in the form $a + ib$, where a and b are real, and write down $(6-5i)(7-2i)$ in a similar form. Hence find the prime factors of

$32^2 + 47^2$.

(b) The equation $x^3 + 2x - 1 = 0$ has roots α, β and γ. Find the monic equations with roots

(i) $-\alpha, -\beta$ and $-\gamma$ **(ii)** $\alpha, -\alpha, \beta, -\beta, \gamma, -\gamma$ **(iii)** α^2, β^2 and γ^2.

(c) The region $\{(x, y) : 0 \leqslant x \leqslant 2 \text{ and } 0 \leqslant y \leqslant 2x - x^2\}$ is rotated about the y-axis. Use the method of slicing to find the volume of the solid formed.

6 A particle moves under gravity in a medium in which the resistance to its motion per unit mass is k times its speed, where k is a constant.

(i) If the particle falls vertically from rest, show that its terminal velocity is given by $V = \frac{g}{k}$.

(ii) If the particle is projected vertically upward with speed V, show that after time t its speed v and height x are given by $v = V(2e^{-kt} - 1)$ and $x = \frac{1}{k}V(2 - 2e^{-kt} - kt)$. Hence show that the greatest height H that the particle can reach is given by $H = \frac{1}{k}V(1 - \ln 2)$.

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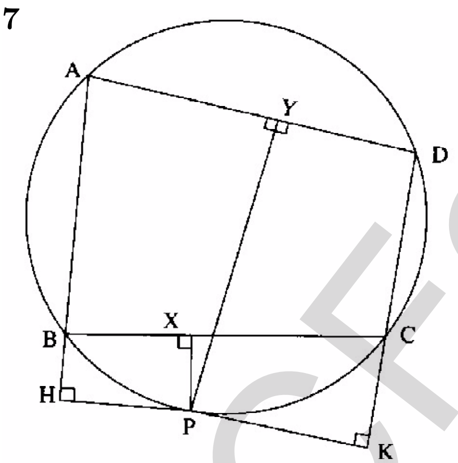


$ABCD$ is a cyclic quadrilateral. P is a point on the circle through A, B, C and D. PH, PX, PK and PY are the perpendiculars from P to AB produced, BC, DC produced and DA, respectively.

(i) Show that $\triangle XPK \| \triangle HPY$

(ii) Hence show that $PX.PY = PH.PK$ and $\frac{PX.PK}{PH.PY} = \frac{XK^2}{HY^2}$.

8 (a) A vertical pole of height 2 m, with base at the point O, stands on the west side of a canal with straight parallel sides running from north to south. Two points A and B both lie on the east side of the canal, A to the north and B to the south of the pole, such that the angle AOB is $150°$. The angles of elevation of the top P of the pole are $45°$ from A and $30°$ from B respectively. Find

(i) the distance AB

(ii) the width of the canal

(b) If six lines are drawn in a plane, no two of which are parallel and no three of which are concurrent, show that there are 15 points of intersection.

(i) If three of these points are chosen at random, find the probability that they all lie on one of the given lines.

(ii) If four of these points are chosen at random, find the probability that they do not all lie on one of the given lines.

PAPER 6

1 (a) For the curve $x^2y^2 - x^2 + y^2 = 0$

(i) Show that $|y| \leqslant 1$ and $|y| \leqslant |x|$

(ii) Find the equations of the asymptotes and the equations of the tangents at the origin. Hence sketch the curve.

(b) (i) Sketch the graph of $y = x^3 - 3px + q$, where $p > 0$ and q are real, showing clearly the coordinates of the turning points.

(ii) Hence show that the roots of the equation $x^3 - 3px + q = 0$ are all real if and only if $q^2 \leqslant 4p^3$.

2 (a) Find $\int \frac{4}{x^2+2x-1}\ dx$

(b) If $a > 0$, use the substitution $u = \frac{1}{x}$ to evaluate $\int_{\frac{1}{a}}^{a} \frac{\ln x}{x^2+1}\ dx$

(c) If $I_n = \int_0^1 (1-x^2)^n\ dx$ for $n \geqslant 0$, show that $I_n = \frac{2n}{2n+1} I_{n-1}$ for $n \geqslant 1$. Hence, or otherwise, show that $I_n = \frac{2^{2n}(n!)^2}{(2n+1)!}$ for $n \geqslant 1$.

3 The point $P(a\sec\theta, b\tan\theta)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is joined to the vertices $A(a, 0)$ and $A'(-a, 0)$. The lines AP and $A'P$ meet the asymptote $y = \frac{bx}{a}$ at Q and R respectively.

(i) Find the coordinates of Q and R.

(ii) Hence find the length QR, showing that it is independent of θ, and show that the area of triangle PQR is $\frac{1}{2}|ab(\sec\theta - \tan\theta)|$ square units.

4 (a) The equation $x^3 + px + q = 0$ has roots α, β and γ. Find the monic cubic equation with roots α^2, β^2 and γ^2.

(b) Use De Moivre's theorem to show that $\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

(i) Find the general solution of $\tan 4\theta = 1$

(ii) Hence find the roots of the equation $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$ in trigonometric form and show that $\tan^2\frac{\pi}{16} + \tan^2\frac{3\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{7\pi}{16} = 28$.

5 (a) (i) Express the complex number $z = -1 + \sqrt{3}i$ in modulus argument form.

(ii) Indicate on an Argand diagram the point P, Q, R and S representing the complex numbers $z, \overline{z}, z^2$ and $\frac{1}{z}$ respectively.

(b) $1 - 2i$ is a root of the equation $z^2 + (2+i)z + k = 0$. Find

(i) the other root

(ii) the value of k

(c) The region $\{(x,y) : 0 \leqslant x \leqslant 2 \text{ and } 0 \leqslant y \leqslant 4x^2 - x^4\}$ is rotated about the y-axis. Use the method of slicing to find the volume of the solid formed.

6 A projectile is fired vertically upward from the surface of the earth with speed V. The acceleration due to gravity is $\frac{gR^2}{x^2}$ where R is the radius of the earth and x is the distance from the centre of the earth.

(i) Neglecting air resistance, show that if $V = \sqrt{gR}$, then the speed v of the projectile at distance x from the centre of the earth is given by $v = \sqrt{gR}\sqrt{\frac{2R-x}{x}}$

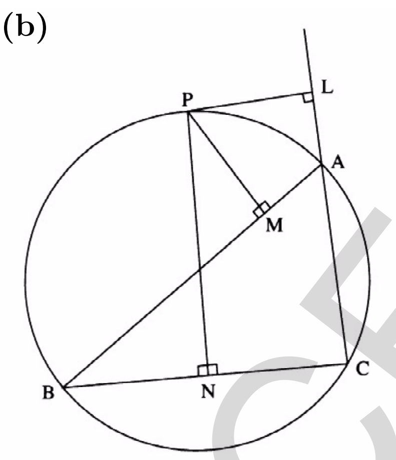
(ii) Hence show that the projectile reaches a height R above the surface of the earth, and find the time taken to reach this height.

7 (a) ADB is a straight line with $AD = a$ and $DB = b$. A circle is drawn on AB as diameter. DC is drawn perpendicular to AB to meet this circle at C.

(i) Show that $\triangle ADC \|!| \triangle CDB$, and hence show that $DC = \sqrt{ab}$

(ii) Deduce geometrically that if $a > 0$ and $b > 0$, then $\sqrt{ab} \leqslant \frac{a+b}{2}$.

(b)



ABC is a triangle inscribed in a circle. P is a point on the minor arc AB. L, M and N are the feet of the perpendiculars from P to CA produced, AB, and BC respectively.

Show that L, M and N are collinear.

8 (a) If $a > 0$, show that $a^2 + \frac{1}{a^2} \geqslant a + \frac{1}{a} \geqslant 2$.

(b) The equation $x^2 - x + 1 = 0$ has roots α and β, and $A_n = \alpha^n + \beta^n$ for $n \geqslant 1$

(i) Without solving the equation, show that $A_1 = 1, A_2 = -1$ and $A_n = A_{n-1} - A_{n-2}$ for $n \geqslant 3$.

(ii) Hence use induction to show that $A_n = 2\cos\frac{n\pi}{3}$ for $n \geqslant 1$.