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NEW SOUTH WALES

Higher School Certificate

Mathematics Extension 2

Exercise 57/67

by James Coroneos*

1. Prove that the equation of the circle described on the line joining (x_1, y_1) and (x_2, y_2) as diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$. Find the equation of the circle having the two points $(-3, -1)$ and $(5, 5)$ as extremities of a diameter. Determine the equations of the lines parallel to $3x - 4y = 0$ on which the circle intercepts a chord of length 8 units.

2. **(i)** Prove that the circles $x^2 + y^2 - 20x - 16y + 128 = 0,\ 4x^2 + 4y^2 + 16x - 24y - 29 = 0$ lie on entirely outside each other, and find the length of the shortest distance from a point on one circle to a point on the other.

(ii) Find the equation of the circle whose centre is at the point $(6, 9)$ and which passes through the centre of the circle $x^2 + y^2 + 4x - 6y - 12 = 0$. If a common tangent touches these circles at P, Q find the length of PQ.

3. Find the equation of the circle which touches the x-axis at the point $(3, 0)$ and passes through the point $(1, 4)$. Prove, if 2θ is the angle between the tangents from the origin to this circle, then $\tan\theta = 5/6$.

4. Two fixed points A and B have coordinates $(-a, 0)$ and $(a, 0)$ respectively. A point A moves in the plane containing the axes of coordinates so that the ratio $PA : PB$ has a constant value λ $(\lambda > 1)$. Show that the locus of P is a circle of centre $\{a(\lambda^2 + 1)/(\lambda^2 - 1), 0\}$ and radius $2a\lambda/(\lambda^2 - 1)$. If the centre of the circle is C, and R is the point in which the circle meets the line AB internally, find the value of λ for which $BC = 2.OR$ where O is the origin.

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5. The equations of the sides of a triangle are $x + y - 4 = 0$, $x - y - 4 = 0$, $2x + y - 5 = 0$. Prove that for all numerical values of p and q the equation $p(x + y - 4)(2x + y - 5) + q(x - y - 4)(2x + y - 5) = (x - y - 4)(x + y - 4)$ represents a curve passing through the vertices of this triangle. Find the values of p and q which make this curve a circle, and so determine the centre and radius of the circumcircle of the triangle.

6. Find the coordinates of the centre and the radius of the circle $x^2 + y^2 - ax - by = 0$. Show that this circle will touch the circle $x^2 + y^2 = c^2$ if $a^2 + b^2 = c^2$, and find the coordinates of the point of contact. Two circles pass through the origin and the point $(1, 0)$ and touch the circle $x^2 + y^2 = 4$. Find the coordinates of the points of contact.

7. Show that the equation $x^2 + y^2 - a^2 + 2\lambda(lx + my + n) = 0$ represents a system of circles through the points of intersection of the circle $C: \ x^2 + y^2 = a^2$ and the line $lx + my + n = 0$. By considering that circle of the system given by the value of λ for which the radius is stationary, or otherwise, find the equation of the circle S which has diameter the chord of the circle C which lies along $lx + my + n = 0$. Show for the circle to exist, then $(l^2 + m^2)a^2 > n^2$. Consider the case when $(l^2 + m^2)a^2 = n^2$. If the circle S cuts the x-axis in two points P, Q; find the value of $OP^2 + OQ^2$, where O is the origin.

8. Prove that the chord joining the points $P_1(2at_1, at_1^2)$ and $P_2(2at_2, at_2^2)$ on the parabola $x^2 = 4ay$ has equation $(t_1 + t_2)x - 2y = 2at_1t_2$. A variable chord of this parabola passes through the fixed point $(0, h)$. Prove that the midpoint of the chord lies on the curve whose equation is $x^2 = 2a(y - h)$. Find the equation of the locus of the point of intersection of the tangents at the ends of this variable chord.

9. Derive the equation of the tangent and normal to the parabola $x^2 = 4ay$ at the point $(2at, at^2)$. The tangents are drawn to this parabola from a point $P(x, y)$ and the normals at the points of contact meet at $Q(\xi, \eta)$. Show that $a\eta + ay = x^2 + 2a^2$ and $a\xi + xy = 0$. What is the locus of P if Q moves on the line **(i)** $y = 2a$ **(ii)** $x = a$?

10. Find the equation of the normal to the parabola $x^2 = 4ay$ at the point $P_1(2at_1, at_1^2)$. The normal at P_1 meets the curve again at $P_3(2at_3, at_3^2)$, find t_3 in terms of t_1. If the normal at P_2 also passes through P_3, prove that $P_1P_2^2 = a^2(t_3^2 + 4)(t_3^2 - 8)$.

11. The coordinates of P, the midpoint of a chord of the parabola $x = 2at, y = at^2$ are (x_0, y_0). Show the gradient of the chord is $x_0/2a$ and give its equation. Find the quadratic equation whose roots are the parameters of the endpoints K_1, K_2 of the chord. Find the equations of the loci of P as the chord is varied in such a way that

(i) it passes through O, the vertex,

(ii) OK_1, OK_2 are perpendicular,

(iii) the tangents at K_1, K_2 meet on the directrix.

In each case describe the locus and its relation to the parabola.

12. Prove that the equation of any parabola whose axis is parallel to Oy is of the form $y = lx^2 + mx + n$. Two parabolas intersect in four points; prove that if their axes are perpendicular, then

(i) the sum of the (directed) distances of the four points from the axis of either parabola is zero, and

(ii) the four points are concyclic.

{*Hint*: *Let the second parabola be* $y^2 = 4ax$}.

13. The focus of a parabola is the point (h, k) and its directrix is $y = d$. Show that its equation can be written in the form $2py = x^2 + 2qx + r$, and express p, q, r in terms of h, k, d. Write down the coordinates of the focus of the parabola $2py = x^2 + 2qx$. The two parabolas $2py = x^2 + 2qx$, $2p'x = y^2 + 2q'y$ have the same focus. Prove that their tangents at the origin are inclined at $45°$.

14. Find the equation of the chord PQ joining the points $(2ap, ap^2)$ and $(2aq, aq^2)$ of the parabola $x^2 = 4ay$. The line through P parallel to the normal at Q meets the line through Q parallel to the normal at P in the point R. Prove that the coordinates of R are $\{a(p+q)(2+pq), -a(2+pq)\}$. If PQ varies so as to pass through the fixed point $K(k, 2a)$, $(k > 2\sqrt{2}a)$ on the line $y = 2a$ find the equation of and describe geometrically the locus of R. Prove in particular that this locus passes through the point where the chord of contact of tangents from K to the parabola $x^2 = 4ay$ meets $y = 2a$.

15. Find the equation of the chord contact of tangents from the point (x_1, y_1) to the parabola $x^2 = 4ay$, and show that it meets the curve in the points $(2at_2, at_2^2)$, $(2at_1, at_1^2)$ where t_1 and t_2 are the roots of the quadratic equation $at^2 - x_1t + y_1 = 0$. The chord PQ of a parabola subtents a right angle at the focus S, and the tangents at P, Q meet at T. The feet of the perpendiculars from T on SP, SQ are M, N respectively. Prove that $TMSN$ is a square.

16. Show the equation of the tangent to parabola $x^2 = 4ay$ at the point $P(2ap, ap^2)$ is $px = y + ap^2$. A triangle ABC has as its sides the tangents to this parabola at the points $P, Q(2aq, aq^2)$, $R(2ar, ar^2)$.

(i) Prove that the centroid (mean centre) of the triangle ABC has coordinates $\{2a(p+q+r)/3, a(pq+qr+pr)/3\}$.

(ii) Find the coordinates of the orthocentre of this triangle, showing that the x-coordinate is independent of p, q and r. {The orthocentre is the point of intersection of the altitudes of the triangles.}

(iii) Prove that the area of the triangle ABC is equal to half the area of the triangle PQR.

17. If a line with gradient m is to be a tangent to both the ellipse $E:\ x^2/a^2 + y^2/b^2 = 1$ and the circle $C:\ x^2+y^2 = r^2$, prove that $m^2 = (r^2-b^2)/(a^2-r^2)$. Hence find the common tangents to the ellipse $x^2/169 + y^2/16 = 1$ and the circle $x^2 + y^2 = 25$.

18. Prove that the line $x\cos\alpha + y\sin\alpha = p$ touches $E:\ x^2/a^2 + y^2/b^2 = 1$ if $p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$ and show that the points of contact are $(a^2\cos\alpha/p, b^2\sin\alpha/p)$. Two points are taken on the minor axis at the same distance from the centre as the foci. Prove that the sum of the squares of the perpendicular distances from these points on to any tangent is constant.

19. The line $y = mx + c$ (where m is fixed and c varies) cuts the ellipse $E:\ x^2/a^2 + y^2/b^2 = 1$ in L, N. Find the coordinates of the midpoint of LN, and hence show that the midpoints of the family of parallel lines lie on a line through the centre O of E with gradient $-b^2/a^2m$. OC, OD are semi-diameters of E such that the product of the gradients of OC, OD is $-b^2/a^2$. If θ, ϕ are the eccentric angles of C, D show that $\phi - \theta = \pi/2$ or $\theta - \phi = \pi/2$.

20. The points P, Q with parameters $\theta, \theta + \frac{1}{2}\pi$ both lie on the ellipse $E:\ x^2/a^2 + y^2/b^2 = 1$. Show that Q has coordinates $(-a\sin\theta, b\cos\theta)$ and prove that $OP^2 + OQ^2 = a^2 + b^2$ (O is the centre of E).

(i) Show that the midpoint M of PQ lies on the ellipse $x^2/a^2 + y^2/b^2 = 1/2$.

(ii) Find the equations of the tangents at P, Q to E and hence obtain the coordinates of T, their point of intersection. Show that T lies on the ellipse $x^2/a^2 + y^2/b^2 = 2$.

(iii) If α is the angle between the tangents at P, Q prove that $\tan\alpha = 2\sqrt{1-e^2}/e^2\sin 2\theta$.

21. P is a variable point with eccentric angle θ on the ellipse $E:\ x^2/16 + y^2/9 = 1$. The tangent at P meets the x, y axes in L, M and the normal at P meets the x, y axes in G, H.

(i) If Q, R are the midpoints of LM, GH respectively, show that the loci of Q, R are curves with equations $16/x^2 + 9/y^2 = 4,\ 64x^2 + 36y^2 = 49$.

(ii) Prove that the minimum length of LM is 7 units.

22. t is any real number; by putting $t = \tan \frac{1}{2}\alpha$, show that the point $\{a(1-t^2)/(1+t^2); 2bt/(1+t^2)\}$ lies on the ellipse $E:\ x^2/a^2 + y^2/b^2 = 1$. If P, Q are the points θ, ϕ on E and PQ passes through the focus $S(ae, 0)$, by finding the gradients of PS, SQ show that $\tan \frac{1}{2}\theta . \tan \frac{1}{2}\phi = \frac{e-1}{e+1}$. Write down the corresponding result if the chord PQ were to pass through the other focus.

23. Prove that the equation of the tangent to the ellipse $E:\ x^2/a^2 + y^2/b^2 = 1$ at the point $P(a\cos\theta, b\sin\theta)$ is $x\cos\theta/a + y\sin\theta/b = 1$. The tangent at a point P on E meets the parabola $y^2 = 4ax$ at the points Q, R and is such that the midpoint of QR lies on the line $y + 2a = 0$. Prove that the product of the perpendiculars from $(a, 0)$ and $(-a, 0)$ on to the tangent is $b^2/2$.

24. P, a point in the first quadrant, lies on $E:\ x^2/a^2 + y^2/b^2 = 1$ and is equidistant from the coordinate axes. Find its coordinates in terms of a, b. The tangent at P meets the x, y axes in points distant $u.v$ from the origin. Prove that the eccentricity of E is $\sqrt{1 - v/u}$.

25. $A'A$ and $B'B$ are the major and minor principal diameters of the ellipse $E:\ x^2/a^2 + y^2/b^2 = 1$. P is the point $(a\cos\theta, b\sin\theta)$. The normal at P meets $A'A$ in M and $B'B$ in N. Prove that $PM : MN = 1 - e^2 : e^2$ where e is the eccentricity.

- **(i)** Show that the limiting position of N as P tends to B is inside or outside E according as $e^2 < \frac{1}{2}$ or $e^2 > \frac{1}{2}$.
- **(ii)** The normal at P meets the curve again in Q. If $e^2 \leq \frac{1}{2}$, deduce that if P is on the arc AB of E then Q is on the arc $A'B'$.

26. P, Q are variable points with eccentric angles $\theta, -\theta$ on the ellipse $E:\ x^2/a^2 + y^2/b^2 = 1$, with major axis AA'.

- **(i)** Find the equations of the tangents at P, Q and hence prove that they meet on the x-axis.
- **(ii)** Show the equation of AP is $b\sin\theta x + a(1-\cos\theta)y = ab\sin\theta$, and find the equation of $A'Q$. Prove that their point of intersection R lies on the hyperbola $H:\ x^2/a^2 - y^2/b^2 = 1$.

27. The ellipse E whose equation is $x^2/a^2 + y^2/b^2 = 1$ has foci F_1 and F_2. Show that the ellipse E' whose equation is $x^2/(a^2+\lambda) + y^2/(b^2+\lambda) = 1$ has the same foci F_1, F_2 as E; (λ is a constant).

- **(a)** Prove the condition for the line $y = mx + c$ to touch E is that $c^2 = a^2m^2 + b^2$. If the tangent to E makes an angle θ with the x-axis, show that its equation is $x\sin\theta - y\cos\theta = \pm\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$.

(b) The line L is a tangent to E, the line M is a tangent to E', and L and M are perpendicular. If L, M make angles θ, ϕ with the x-axis ($\phi > 0$), show that $\phi = \pi/2 + \theta$, and prove that L, M meet on the circle $x^2+y^2 = a^2+b^2+\lambda$. {*Hint*: *Square and add corresponding sides of the tangent equations.*}

28. P is the point $(a\cos\theta, b\sin\theta)$ and Q is the point $(a\sec\theta, b\tan\theta)$. Sketch on the same diagram the curves which are the loci as θ varies of P and Q. Mark on them the range of positions occupied by P and Q respectively as θ varies from $\pi/2$ to π.

(i) Prove that for any value of θ, the line PQ passes through one of the common points of the two curves.

(ii) Prove also that the tangent to the first curve at P meets the tangent to the second curve at Q in a point on the common tangent to the two curves at their other common point.

29. **(i)** The line $y = mx+c$ cuts the hyperbola $H:\ x^2/a^2-y^2/b^2 = 1$ in P and Q. Show that the midpoint of PQ lies on the diameter $y = b^2x/a^2m$.

(ii) Show that the line $y - mx = \pm\sqrt{a^2m^2-b^2}$ touches H for all values of m.

(a) Write down the equation of the perpendicular from $S(ae, 0)$ to this tangent. Hence prove that the foot of the perpendicular from S to any tangent to H lies on the auxiliary circle. {*Hint*: *Square and add corresponding sides of the above equations.*}

(b) Show also that perpendicular tangents to H meet on the circle $x^2+y^2 = a^2-b^2$. {*Hint*: *Square the equation of the tangent and write as a quadratic equation in* m}

30. **(i)** The line $y = mx+c$ is a tangent to the hyperbola $H:\ x^2/16-y^2/9 = 1$. Prove that $c^2 = 16m^2-9$. The tangents from $P(X, Y)$ to H meet at right-angles. Prove that P lies on the circle $x^2+y^2 = 7$. {*Hint*: *See method in question* **29 (ii) (b)**}

(ii) Show that the line $x\cos\alpha + y\sin\alpha = p$ touches the hyperbola $H:\ x^2/a^2 - y^2/b^2 = 1$ if $p^2 = a^2\cos^2\alpha - b^2\sin^2\alpha$. Perpendiculars are drawn from the foci of H on to any tangent. Prove that the product of their lengths is independent of the position of the tangent.

31. $P(a\sec\theta, b\tan\theta)$ lies on $H:\ x^2/a^2 - y^2/b^2 = 1$. If S, S' are the foci, prove that $SP = a(e\sec\theta - 1)$, $S'P = a(e\sec\theta + 1)$.

(i) C is the centre of the rectangular hyperbola $h:\ x^2 - y^2 = a^2$, whose foci are F, F'. $Q(a\sec\phi, a\tan\phi)$ lies on h. Show the eccentricity of h is $\sqrt{2}$ and prove $FQ.F'Q = CQ^2$.

(ii) Show that the foci s, s' of the hyperbola $x^2/16 - y^2/8 = 1$ have coordinates $(\pm 2\sqrt{6}, 0)$. A is the point $(4\sec\theta, 2\sqrt{2}\tan\theta)$, where $0 < \theta < \pi/2$, and AB is a diameter such that AB subtends right-angles at s, s'. Prove that $\theta = \pi/6$. Hence find the area and the perimeter of the rectangle $asBs'$.

32. The points P, Q with eccentric angles $\theta, \pi/2 + \theta$ lie on the hyperbola H : $x^2/a^2 - y^2/b^2 = 1$. Show that if P has coordinates $(a\sec\theta, b\tan\theta)$ then Q has coordinates $(-a\operatorname{cosec}\theta, -b\cos\theta)$. Prove that the tangent at P has equation $bx - ay\sin\theta = ab\cos\theta$ and write down the equation of the tangent at Q. Show that the tangents at P, Q meet on the hyperbola $2x^2/a^2 - y^2/b^2 = 1$. Express the eccentricity of this hyperbola in terms of e, the eccentricity of H.

33. F is the point $(ae, 0)$ and d is the line $x = a/e$ $(e > 1)$. M is the foot of the perpendicular from a variable point P to d, and P moves so that $FP^2 = e^2.PM^2$. Find the equation of the locus of P.

(i) Draw a sketch showing clearly the principal axes of the curve, the foci, the directrices and the asymptotes, marking on each of them its equation or coordinates. Express e in terms of α, the angle between the asymptotes.

(ii) A line, parallel to one asymptote, is drawn through a focus F of a hyperbola. It meets the hyperbola in H, the directrix corresponding to F in D, the other asymptote in K and the conjugate axis in R. Prove that $FH = HD$ and $FK = KR$.

(iii) The tangent at $P(x_1, y_1)$ to the hyperbola meets the directrix corresponding to F in L. Prove that FL is perpendicular to FP.

34. Prove that the gradient of the chord joining the points (x_1, y_1) and (x_2, y_2) on the hyperbola $H:\ x^2/a^2 - y^2/b^2 = 1$ is $b^2(x_1 + x_2)/a^2(y_1 + y_2)$. Hence, or otherwise, prove that the equation of the chord of the hyperbola whose midpoint is (x_0, y_0) is $(x - x_0)x_0/a^2 - (y - y_0)y_0/b^2 = 0$. Find also the locus of the midpoints of the chords of the hyperbola which are parallel to the fixed line $y = mx$.

35. The perpendicular from the origin O to the tangent at a point $P(cp, c/p)$ on the rectangular hyperbola $xy = c^2$ meets the curve at Q and R. The chords PQ and PR meet the x-axis at U and V. Prove that the midpoint of UV is the foot of the perpendicular from P to the x-axis.

36. The tangent at P to the rectangular hyperbola $xy = c^2$ meets the lines $x - y = 0$ and $x + y = 0$ at A and B, and Δ denotes the area of the triangle OAB where O is the origin. The normal at P meets the x-axis at C and the y-axis at D. If Δ_1 denotes the area of the triangle ODC, show that $\Delta^2\Delta_1 = 8c^6$.

37. **(i)** The gradients of the tangents to the parabola $y^2 = 4ax$ and the rectangular hyperbola $xy = c^2$ at their point of intersection are m_1, m_2 respectively. Prove that $m_2 = -2m_1$.

(ii) The normal at a variable point P on the curve $xy = c^2$ meets the asymptotes in Q, R. Show that the locus of the midpoint of QR has equation $4x^3y^3 + c^2(x^2 - y^2)^2 = 0$.

38. The points P_i $(i = 1, 2, 3)$ lie on the rectangular hyperbola $x = ct, y = c/t$ and have the parameters t_i $(i = 1, 2, 3)$. Show that the line through P_1 perpendicular to P_2P_3 has equation $t_1t_2t_3x - t_1y + c(1 - t_1^2t_2t_3) = 0$. Deduce that the orthocentre H of the triangle $P_1P_2P_3$ is the point with parameter $-1/t_1t_2t_3$. Show also that, in the special case, when P_2P_3 are the vertices of the hyperbola, then the lines joining the origin to P_1 and H are equally inclined to the x, y axes respectively.

39. Show that the tangents at the points $P(cp, cp^{-1})$ and $Q(cq, cq^{-1})$ to $xy = c^2$ meet at the point $T\{2cpq(p + q)^{-1}, 2c(p + q)^{-1}\}$.

(i) Write down the equation of the other tangent to $xy = c^2$ parallel to the tangent to $xy = c^2$ at P.

(ii) Two pairs of parallel tangents to $xy = c^2$ form a parallelogram. Prove that if one pair of opposite vertices lies on $xy = k^2$, the other pair lies on $xy = a$ and find the value of a in terms of c and k. Discuss the possible range of values of k.

40. Find in its simplest form the condition that the four points with coordinates $(kt, k/t)$, where the parameter t has the values a, b, c, d should lie on a circle. A, B, C, D are four points on the rectangular hyperbola $xy = c^2$ and are not concyclic. If the circles BCD, CAD, ABD, ABC meet this hyperbola again in the points $\alpha, \beta, \gamma, \delta$ respectively, prove that the middle points of the chords $A\alpha, B\beta, C\gamma, D\delta$ lie on another rectangular hyperbola with the same asymptotes.

41. **(i)** Show that the equation of the chord joining the 2 points (x_1, y_1) and (x_2, y_2) on the rectangular hyperbola $xy = c^2$ is
$x/(x_1 + x_2) + y/(y_1 + y_2) = 1$.

(ii) Prove that the normal at P to the hyperbola $xy = c^2$ has equation $xx_1 - yy_1 = x_1^2 - y_1^2$. If the normal at P meets the curve again at $Q(x_2, y_2)$ show that $x_1x_2 = -y_1^2$ and $y_1y_2 = -x_1^2$. {*P has coordinates* (x_1, y_1)}

42. Show that the tangent at the point $P(x_1, y_1)$ on the curve $xy = c^2$ is $xy_1 + x_1y = 2c^2$ and that the tangent at P on the curve $x^2 - y^2 = a^2$ is $xx_1 - yy_1 = a^2$. P is a point of intersection of the rectangular hyperbolas $xy = c^2$, $x^2 - y^2 = a^2$. The tangent at P to $xy = c^2$ meets its asymptotes in A, C and the tangent at P to $x^2 - y^2 = a^2$ meets its asymptotes in B, D. Prove that $ABCD$ is a square.

43. **(i)** The normal at $P(ct, c/t)$ on the hyperbola $xy = c^2$ meets the line $y = x$ at G. Find the length of PG in terms of t, and show that there is no point on the hyperbola for which PG is less than $c\sqrt{2}$. {*Hint*: *Use calculus.*}

(ii) Show that the straight lines joining the origin O to the points of intersection of the line $y = mx + c$ and the hyperbola $xy = c^2$ cannot be at right angles. If α is the angle between these lines, find $\tan\alpha$ in terms of m, and prove it is a maximum when m is the positive root of the equation $3m^2 + m - 1 = 0$.