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NEW SOUTH WALES

Higher School Certificate

Mathematics Extension 2

Exercise 53/67

by James Coroneos*

1. P is the point $(3\cos\theta, 2\sin\theta)$. Show clearly the positions of P and the angles θ when $\theta = \pi/4$ and when $\theta = 2\pi/3$. What is the cartesian equation of the locus of P? On the sketch of the positions of P draw this locus.

(i) Show that the equation of the tangent at P is $2x\cos\theta + 3y\sin\theta = 6$. If this tangent meets the major and minor axes of the ellipse in T, R respectively, prove that the area of $\triangle ORT$ is $6\text{cosec}\, 2\theta$. (O is the origin.)

(ii) Show that the normal at P has equation $3x\sin\theta - 2y\cos\theta = 5\sin\theta\cos\theta$. If this normal meets the x, y axes at G, H respectively prove that the maximum area of $\triangle OGH$ is $25/24$ unit^2. Where is P for this maximum area?

(iii) Show that the tangents at the points where the line $x = 2y$ meets the ellipse have equations $8x + 9y = \pm 30$. {*Use parameters.*}

2. **(i)** P, Q are points with eccentric angles α, β on the ellipse $x = 6\cos\theta, y = 5\sin\theta$. Find the gradient of the tangent at P to the ellipse, and show that the tangents at P, Q are perpendicular if $\tan\alpha . \tan\beta = -25/36$.

(ii) Find the coordinates of the points on the ellipse $x = \sqrt{3}\cos\phi, y = \sin\phi$ where the tangent makes equal intercepts with the major and minor axes.

3. Find the eccentricity, the coordinates of the foci S, S' and the equations of the directrices of the ellipse $x^2/4 + y^2/3 = 1$, whose centre is O. Sketch the curve. State the coordinates of A, A' the ends of the major axis and of B, B' the ends of the minor axis.

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*Other resources by James Coroneos are available. Write to P.O. Box 25, Rose Bay, NSW, 2029, Australia, for a catalogue. Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX.

(i) P is the point $(2\cos\alpha, \sqrt{3}\sin\alpha)$ and PM, Pm are drawn to meet the major, minor axes at right-angles in M, m respectively. Show that $MP^2 : A'M.MA = 3 : 4$ and $mP^2 : B'm.mB = 4 : 3$.

(ii) If OP cuts the directrix corresponding to S in Q, show that SQ is perpendicular to the tangent at P.

(iii) Show that $SP = 2 - \cos\alpha$ and hence prove that $SP + S'P = 4$

(iv) Prove that the normal at P has equation $2x\sec\alpha - \sqrt{3}y\text{cosec}\,\alpha = 1$. If this normal cuts the major axis in G, show that the ratio $GS : SP$ is the same for all points on the ellipse.

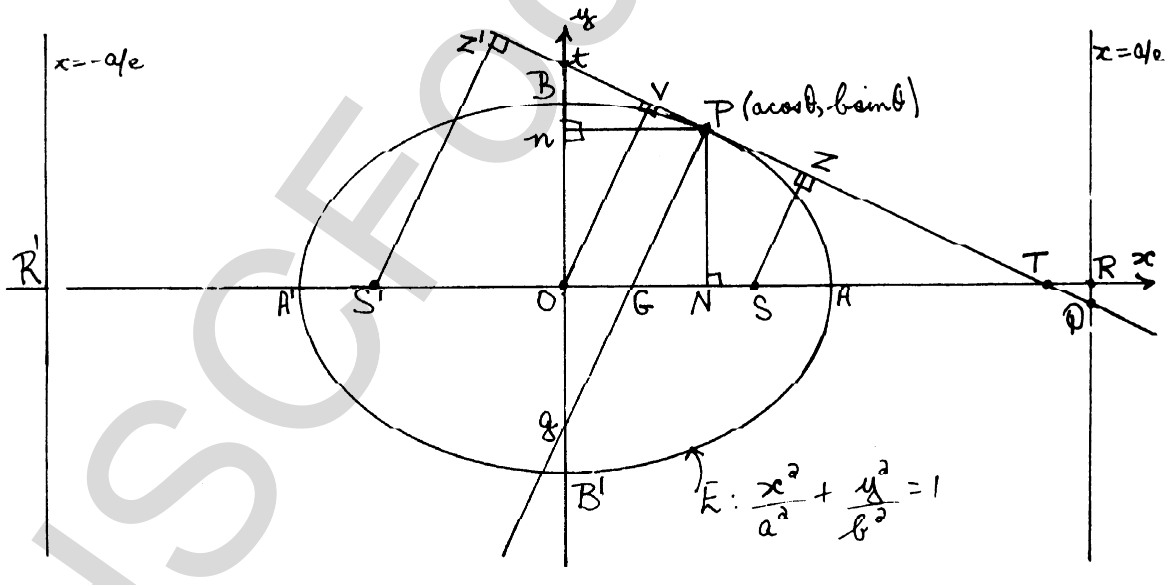
(v) Prove that the normal bisects the angle $S'PS$.

4. Find the equation of the tangent at the point $P(x_1, y_1)$ on the ellipse $x^2/8 + y^2/3 = 1$ and show that the normal at P has equation $8xy_1 - 3yx_1 = 5x_1y_1$. This tangent meets the x, y axes at T, R respectively whilst the normal meets these axes at G, H respectively. PN is perpendicular to the x-axis. Prove that **(i)** $|OH.OR| = 5$ **(ii)** $ON.OT = 8$ **(iii)** $PN.OR = 3$

5. For an ellipse E with eccentricity e, where a, b are the major and minor semi-axes, simplify the following:

(i) $\frac{a^2\sin^2\theta + b^2\cos^2\theta}{1 - e^2\cos^2\theta}$ **(ii)** $\sqrt{(a\cos\theta - ae)^2 + (b\sin\theta)^2}$

6.



In the diagram, P is the point $(a\cos\theta, b\sin\theta)$ on the ellipse $E:\ x^2/a^2 + y^2/b^2 = 1$, which has centre O and foci S, S'. PT and PG are the tangent and normal at P, meeting the axes at T, t and G, g. PN, Pn are perpendicular to the axes; A, A', B, B' are the vertices of E. $SZ, OV, S'Z'$

are perpendiculars from S, O, S' respectively to the tangent PT. PT cuts the directrix $x = a/e$ in Q. Prove the following results:

(i) The equation of the tangent PT is $bx\cos\theta + ay\sin\theta = ab$, and of the normal PG is $ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$.

(ii) $ON.OT = a^2$ **(iii)** $On.Ot = b^2$ **(iv)** $PN^2 : A'N.NA = b^2 : a^2$

(v) $SB = S'B = a$ **(vi)** $OG = e^2.ON$ **(vii)** $Og = -a^2e^2.On/b^2$

(viii) $SG = e(a - e.ON)$ **(ix)** $S'G = e(a + e.ON)$ **(x)** $SZ.S'Z' = b^2$

(xi) $PG.OV = b^2$ **(xii)** Tg and tG are at right-angles **(xiii)** $P\hat{S}Q = 90°$

(xiv) $SP = a(1-e\cos\theta)$ and $S'P = a(1+e\cos\theta)$. Hence show that $SP+S'P$ is independent of the position of P on E.

(xv) $SG.SP$ and $S'G = e.S'P$. Hence deduce that the normal at P bisects the angle $S'PS$.

{In the above diagram, if P is (x_1, y_1) show that the tangent and normal at P have equations $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$, $\frac{xy_1}{b^2} - \frac{yx_1}{a^2} = \frac{(a^2-b^2)x_1y_1}{a^2b^2}$ respectively. Hence prove the results **(ii)**-**(xiii)**.}

7. If λ is any positive number, show that the equation $x^2/a^2 + y^2/b^2 = \lambda^2$ gives a family of ellipses whose eccentricities are independent of λ. What happens if $\lambda = 0$? P is any point $(a\cos\theta, b\sin\theta)$ on the ellipse $E:\ x^2/a^2 + y^2/b^2 = 1$, whose centre is O. OP (produced if necessary) meets the ellipse $x^2/a^2 + y^2/b^2 = \lambda^2$ in Q. Prove that $OQ = \lambda.OP$.

8. P is the point $(a\cos\theta, b\sin\theta)$ on the ellipse E and Q is the corresponding point on the auxiliary circle C (i.e., P, Q have the same abscissa). State the coordinates of Q.

(i) Find the equation of the tangent at P to E and at Q to C. Prove that these tangents meet on the major axis of the ellipse.

(ii) Show that the perpendicular distance from a focus S of E to the tangent at Q to C is equal to SP.

(iii) Find the equation of OQ and of the normal to E at P. Show that these meet on the circle $x^2 + y^2 = (a + b)^2$.

9. For the ellipse $E:\ x^2/a^2 + y^2/b^2 = 1$, show the length of the latus rectum is $2b^2/a$ units. The latus rectum through the focus S meets E at G (G is in the first quadrant), find the equation of the tangent to E at G. Show that the tangent intersects the major axis of E where the directrix does and intersects the minor axis where the auxiliary circle does. Also prove that the normal to E at G passes through an end of the minor axis if $e^4 + e^2 = 1$.

10. P is the point $(a\cos t, b\sin t)$ on the ellipse $E:\ x^2/a^2 + y^2/b^2 = 1$.

(i) If θ is the angle between the tangent at P and the line SP (where S is a focus), show that $\tan\theta = \pm b/ae\sin t$.

(ii) Show that the normal at P has equation $ax\sec t - by\operatorname{cosec} t = a^2 - b^2$. This normal cuts the major axis in G and the minor axis in N. If S' is the other focus of S, prove that
(a) $P^2 = (1-e^2).SP.S'P$ **(b)** $NS^2 : SP.S'P = a^2 - b^2 : b^2$ where e is the eccentricity of E.

11. SM is the perpendicular from the focus $S(ae, 0)$ to the tangent at $P(a\cos\theta, b\sin\theta)$ on the ellipse $E:\ x^2/a^2 + y^2/b^2 = 1$.

(i) Find an expression for $\sin S\hat{P}M$ and deduce that $SP, S'P$ make equal angles with the tangent, and further that the normal at P bisects the angle $S'PS$.

(ii) If SM meets the directrix corresponding to S in R, show that the points O, P, R are collinear.

12. **(i)** Show that the chord joining the points with eccentric angles $\alpha+\beta, \alpha-\beta$ on the ellipse $E:\ x^2/a^2 + y^2/b^2 = 1$ is parallel to the tangent at the point with eccentric angle α.

(ii) P, Q are points $\theta, \theta+\gamma$ on $E:\ x^2/a^2+y^2/b^2 = 1$. Show that the length of the perpendicular from Q to OP is $|ab\sin\gamma/\sqrt{b^2\sin^2\theta + a^2\cos^2\theta}|$, and hence prove the area of the triangle OPQ is independent of θ. (O is the origin.)

13. Show that the tangent at $(a\cos\theta, b\sin\theta)$ on $E : x^2/a^2 + y^2/b^2 = 1$ has equation $b\cos\theta x + a\sin\theta y - ab = 0$. Prove that the product of the perpendiculars from $(c, 0)$ and $(-c, 0)$ to this tangent is equal to b^2 if $c^2 = a^2 - b^2$. If p, q are the perpendiculars from $(\pm ae, 0)$ to the tangent, and if it is given that $p^2 + q^2 = 2(a^2 - b^2)$, show that $e^2 = 1/(1 + \sin^2\theta)$ and hence deduce that e is not less than $1/\sqrt{2}$. (e is the eccentricity of E)

14. **(i)** P is any point on the ellipse $E:\ x^2/a^2 + y^2/b^2 = 1$. The diagram through P cuts the directrix in L. The perpendicular from S, the corresponding focus, to OL cuts this directrix in M. Prove that the product of the gradients of OL, OM is $-b^2/a^2$.

(ii) Find the ratio $a : b$ for which the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the parabola $y^2 = 4ax$ cut at right angles.

15. The ellipse $E:\ x^2/16+y^2/9 = 1$ cuts the minor axis at B, B'. P is the point $(4\cos\theta, 3\sin\theta)$. Show that the tangent at P to E has equation $3x\cos\theta + 4y\sin\theta = 12$. If this tangent cuts the tangents at B, B' in C, C' respectively show that $CB.C'B' = 16$.

(i) The circle on CC' as diameter meets the major axis of E in D, D'. Prove that $OD.OD' = 7$.

(ii) Show that CC' cannot subtent a right-angle at any point on the minor axis.

16. S is the focus $(ae, 0)$ of the ellipse $E: \; x^2/a^2 + y^2/b^2 = 1$ and P is the point $(a\cos\theta, b\sin\theta)$.

(i) Find the coordinates of the centre C and the radius of the circle on SP as diameter. Determine the distance OC and hence show that the circle on diameter SP touches the auxiliary circle.

(ii) Show that as P moves on E, the midpoint of SP always lies on the ellipse $(\frac{2x}{a} - e)^2 + (\frac{2y}{b})^2 = 1$. Show the centre of this ellipse is midway between O, S and that it has the same eccentricity as E.

17. P is the point $(5\cos\phi, 4\sin\phi)$ on the ellipse $x^2/25 + y^2/16 = 1$. Find the equation of the tangent at P and the coordinates of the focus F (on the positive x-axis). If FM is the perpendicular distance from F to the tangent at P, and OQ is the semi-diameter parallel to this tangent, show that $FP : FM = OQ : 4$.

18. Show that the line $y = mx = \pm\sqrt{a^2m^2 + b^2}$ touches the ellipse $E: \; x^2/a^2 + y^2/b^2 = 1$ for all values of m. Write down the equation of the perpendicular from a focus to this tangent and hence show the foot of this perpendicular lies on the auxiliary circle. {*Hint*: *Square and add corresponding sides of these* 2 *equations.*}

19. **(i)** If the line $y = mx + c$ is a tangent to the ellipse $x^2/6 + y^2/3 = 1$ show that $c^2 = 6m^2 + 3$. The tangents to this ellipse from a point $P(X, Y)$ meet at right angles. Prove the locus of P is the circle $x^2 + y^2 = 9$. {*Hint*: *Square the equation of the tangent and write as a quadratic in* m.}

(ii) Find the condition for the line $y = mx + c$ to touch the circle $x^2 + y^2 = 9/2$ and the ellipse $x^2/6 + y^2/3 = 1$. Hence obtain the equations of the four common tangents to these two curves.

20. Find the equation of the tangent at the point $P(a\cos\phi, b\sin\phi)$ to the ellipse $E: \; x^2/a^2 + y^2/b^2 = 1$, whose centre is O. QQ' is the diameter of E parallel to this tangent and the perpendicular from the focus $S(ae, 0)$ to the tangent at P meets it in M. If the ellipse meets the y-axis in B, B' show that $SP : SM = OQ : OB$.