NEW SOUTH WALES

Higher School Certificate

Mathematics Extension 2

Exercise 47/67

by James Coroneos*

1. Starting from the identity $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$, find an expression for $\cos 4\theta$ in terms of powers of $\cos\theta$. Write down an expression for $\sin 4\theta$ in terms of $\sin\theta, \cos\theta$ and hence determine a result for $\tan 4\theta$ in terms of $\tan\theta$.

2. Express $\cos 5\theta$ as a polynomial in $\cos\theta$, and obtain a similar expression for $\sin 5\theta$ as a polynomial in $\sin\theta$.

(a) Solve the equation $\cos 5\theta = 1$ for $0 \le \theta < 2\pi$, and hence show that the roots of the equation $16x^5 - 20x^3 + 5x - 1 = 0$ are $x = \cos\frac{2r\pi}{5}$ for $r = 0, 1, 2, 3, 4$. Hence prove that $\cos\frac{\pi}{5}\cos\frac{2\pi}{5} = \frac{1}{4}$ and $\cos\frac{\pi}{5} - \cos\frac{2\pi}{5} = \frac{1}{2}$.

(b) Solve the equation $\cos 5\theta = 0$ for $0 \le \theta < 2\pi$, and hence show that the roots of the equation $16x^4 - 20x^2 + 5 = 0$ are $x = \cos\frac{(2r+1)\pi}{10}$ for $r = 0, 1, 3, 4$. Determine the exact value for $\cos\frac{\pi}{10}\cos\frac{3\pi}{10}$ and $\cos^2\frac{\pi}{10} + \cos^2\frac{3\pi}{10}$.

(c) Solve the equation $\sin 5\theta = 1$ for $0 \le \theta < 2\pi$, and hence show that the roots of the equation $16x^4 + 16x^3 - 4x^2 - 4x + 1 = 0$ are $x = \sin\frac{(4r+1)\pi}{10}$, where $r = 0, 2, 3, 4$. Determine the exact value of $\sin\frac{\pi}{10}\sin\frac{3\pi}{10}$.

(d) Show that $x = \pm\sin\frac{r\pi}{5}$ for $r = 1, 2$ are the roots of the equation $16x^4 - 20x^2 + 5 = 0$, and prove that $\sin^2\frac{\pi}{5} + \sin^2\frac{2\pi}{5} = \frac{5}{4}$.

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*Other resources by James Coroneos are available. Write to P.O. Box 25, Rose Bay, NSW, 2029, Australia, for a catalogue. Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX.

3. Prove that $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$

(i) Hence show that $x = \cos\frac{(2r+1)\pi}{14}$ for $r = 0, 1, 2, 4, 5, 6$ are the roots of the equation $64x^6 - 112x^4 + 56x^2 - 7 = 0$. Deduce that $\cos\frac{\pi}{14}.\cos\frac{3\pi}{14}.\cos\frac{5\pi}{14} = \frac{\sqrt{7}}{8}$ and $\cos^2\frac{\pi}{14} + \cos^2\frac{3\pi}{14} + \cos^2\frac{5\pi}{14} = \frac{7}{4}$

(ii) Solve the equation $\cos 7\theta = 1$ for $0 \le \theta < 2\pi$, and hence show that $x = \cos\frac{2k\pi}{7}$ for $k = 0, 1, 2, 3, 4, 5, 6$ are the roots of the equation $64x^7 - 112x^5 + 56x^3 - 7x = 1$. Deduce that $\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2}$ and $\cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{5\pi}{7} = -\frac{1}{8}$.

4. If $z = \cos\theta + i\sin\theta$, show that $z + z^{-1} = 2\cos\theta$ and $z^n + z^{-n} = 2\cos n\theta$. Find corresponding expressions for $z - z^{-1}$ and $z^n - z^{-n}$.

(i) By expanding $(z + \frac{1}{z})^4$, find an expression for $\cos^4\theta$ in the form $a\cos 4\theta + b\cos 2\theta + c$. Check this result when $\theta = \pi/4$ and evaluate $\int_0^\pi \cos^4\theta\ d\theta$.

(ii) Use the expansion of $(z - \frac{1}{z})^5$ to express $\sin^5\theta$ in the form $a\sin 5\theta + b\sin 3\theta + c\sin\theta$, and state the values of a, b, c.

5. Prove that $\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$, and hence evaluate $\int_0^{\pi/2} \cos^6\theta\ d\theta$.

6. Show that $(1 + \cos 2\theta + i\sin 2\theta)^n = 2^n\cos^n\theta(\cos n\theta + i\sin n\theta)$; where n is a positive integer. By considering the binomial expansion of $(1+z)^n$ where $z = \cos 2\theta + i\sin 2\theta$, show that $2^n\cos^n\theta\cos n\theta = \sum_{r=0}^{n} {}^nC_r\cos 2r\theta$, and hence evaluate $\int_0^{\pi/2}\cos^n\theta\cos n\theta\ d\theta$.

7. Prove that $1 - \cos\theta - i\sin\theta = 2\sin\frac{\theta}{2}\{\cos(\frac{\pi-\theta}{2}) - i\sin(\frac{\pi-\theta}{2})\}$. From the binomial expansion of $(1-x)^n$ where $x = \cos\theta + i\sin\theta$ and n is a positive integer, show that $2^n\sin^n\frac{\theta}{2}\cos n(\frac{\pi-\theta}{2}) = \sum_{r=0}^{n}(-1)^r {}^nC_r\cos r\theta$, and obtain a similar expression for $2^n\sin^n\frac{\theta}{2}\sin n(\frac{\pi-\theta}{2})$.

8. Show that $1 - \cos 2\theta + i\sin 2\theta = 2i\sin\theta(\cos\theta - i\sin\theta)$. Use the binomial expansion of $(1 - \cos 2\theta + i\sin 2\theta)^{2n}$, where n is an even positive integer, to prove that $2^{2n}\sin^{2n}\theta\cos 2n\theta = \sum_{r=0}^{2n}(-1)^r {}^{2n}C_r\cos 2r\theta$, and hence evaluate $\int_0^\pi \sin^{2n}\theta\cos 2n\theta\ d\theta$.

9. By substituting $z = r(\cos\theta + i\sin\theta)$ in the identity $1 + z + z^2 + \cdots + z^n + \cdots = \frac{1}{1-z}$, where $|z| < 1$, prove that $1 + r\cos\theta + r^2\cos 2\theta + r^3\cos 3\theta + \cdots = \frac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2}$ and $r\sin\theta + r^2\sin 2\theta + r^3\sin 3\theta + \cdots = \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}$.

10. If $C = 1 + \cos\theta + \cos 2\theta + \cos 3\theta + \cdots$ and $S = \sin\theta + \sin 2\theta + \sin 3\theta + \cdots$. Show that $C + iS = \frac{1}{1 - \cos\theta - i\sin\theta}$, and hence prove that $C = \frac{1}{2}$ and $S = \frac{1}{2}\cot\frac{\theta}{2}$.

11. If $X = 1 + \frac{a}{b}\cos C + (\frac{a}{b})^n \cos 2C + (\frac{a}{b})^3 \cos 3C + \cdots$ and $Y = \frac{a}{b}\sin C + (\frac{a}{b})^2 \sin 2C + (\frac{a}{b})^3 \sin 3C + \cdots$, where $|a| < |b|$ show that $X + iY = \frac{b(b - a\cos C + ia\sin C)}{(b - a\cos C)^2 + a^2 \sin^2 C}$. If A, B, C are the angles and a, b, c the sides of a triangle ABC, hence prove $X = b\cos A/c$ and $Y = b\sin A/c$. {*Hint: You may assume in* $\triangle ABC$ *that* $b = a\cos C + c\cos A$.}

12. Express the complex number $1 + i$ in the form $r(\cos\theta + i\sin\theta)$, and hence prove that $(1+i)^n + (1-i)^n = 2(2^{n/2}.\cos\frac{n\pi}{4})$ where n is a positive integer. If $(1+x)^n = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n$, prove that $p_0 - p_2 + p_4 - \cdots = 2^{n/2}\cos\frac{n\pi}{4}$ and $p_1 - p_3 + p_5 - \cdots = 2^{n/2}\sin\frac{n\pi}{4}$

13. If $u = z + \frac{1}{z}$, show that $z^6 + \frac{1}{z^6} = u^6 - 6u^4 + 9u^2 - 2$. Hence prove that $1 + \cos 12\theta = 2(32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1)^2$.

14. By using de Moivre's theorem, prove that $\frac{\sin 7\theta}{\sin\theta} = -64\sin^6\theta + 112\sin^4\theta - 56\sin^2\theta + 7$. Show that $8\cos\theta\cos 2\theta\cos 3\theta - 1 = \frac{\sin 7\theta}{\sin\theta}$, and find all the angles θ between 0 and 2π for which $\cos\theta\cos 2\theta\cos 3\theta = \frac{1}{4}$.

15. Use de Moivre's theorem to obtain an expression for $\tan 5\theta$ in terms of $\tan\theta$. Show that $\pm\tan\frac{r\pi}{5}$, $r = 1, 2$ are the roots of the equation $t^4 - 10t^2 + 5 = 0$ and deduce that $\tan\frac{\pi}{5}\tan\frac{2\pi}{5} = \sqrt{5}$ and $\tan^2\frac{\pi}{5} + \tan^2\frac{2\pi}{5} = 10$.

16. Use de Moivre's theorem to obtain expressions for $\cos 6\theta$, $\sin 6\theta$ in terms of $\sin\theta$ and $\cos\theta$.

(i) Prove that $\cot 6\theta = \frac{1 - 15t^2 + 15t^4 - t^6}{2t(3 - 10t^2 + 3t^4)}$ where t denotes $\tan\theta$.

(ii) Give all the roots of $\cos 6\theta = 0$ for $0 \leq \theta < 2\pi$, and hence find the value of $\tan\frac{\pi}{12}\tan\frac{5\pi}{12}$ and $\tan^2\frac{\pi}{12} + \tan^2\frac{5\pi}{12}$.

17. By expanding $(\cos\theta + i\sin\theta)^6$ in 2 different ways, express $\cos 6\theta$ in a series of powers of $\cos\theta$. Find the roots of $\cos 6\theta = 0$ and hence prove that $\cos 6\theta = 32(\cos^2\theta - \cos^2\frac{\pi}{12})(\cos^2\theta - \frac{1}{2})(\cos^2\theta - \cos^2\frac{5\pi}{12})$

18. Establish the following results, where n is any positive integer greater than 1: **(i)** $(\cot\theta + i)^n - (\cot\theta - i)^n = 2i\sin n\theta/\sin^n\theta$

(ii) the roots of the equation $(x + i)^n = (x - i)^n = 0$ are $\cot\frac{\pi}{n}, \cot\frac{2\pi}{n}, \cot\frac{3\pi}{n}, \ldots, \cot\frac{(n-1)\pi}{n}$.

(iii) $(x + i)^n - (x - i)^n = 2i\{{}^nC_1 x^{n-1} - {}^nC_3 x^{n-3} + {}^nC_5 x^{n-5} - \cdots\}$

(iv) $\cot^2\frac{\pi}{n} + \cot^2\frac{2\pi}{n} + \cot^2\frac{3\pi}{n} + \cdots + \cot^2\frac{(n-1)\pi}{n} = \frac{(n-1)(n-2)}{3}$ {*Use the results in* **(ii)**, **(iii)**.}