NEW SOUTH WALES

Higher School Certificate

Mathematics Extension 2

Exercise 12/67

by James Coroneos*

1. If $z = x + iy$, express $Z = \frac{z-1}{z}$ in the form $X + iY$ (where X, Y are real). Hence or otherwise, prove that if the point representing z on the Argand diagram describes a unit circle about the origin, then the point representing Z also describes a unit circle, and find the centre of this circle.

2. Prove that if the ratio $\frac{z-1}{z-i}$ is
 (a) purely imaginary, the point z lies on the circle whose centre is at the point $\frac{1}{2}(1+i)$ and whose radius is $1/\sqrt{2}$.
 (b) real, the point z lies on the line through the points $1, i$.

3. If $|z| = 1$, and Q represents the complex number Z, find the locus of Z if **(i)** $Z = 3z$ **(ii)** $Z = z + 3$ [Hint: start $z = Z - 3$] **(iii)** $Z = 4z + 9$.

4. **(i)** Two points P, Q represent the complex numbers z, $2z + 3 + i$ respectively. If P moves on the circle $|z| = k$, how does Q move?
 (ii) If the real part of $\frac{z-4}{z-2i}$ is zero, prove that the locus of the point representing z in the Argand diagram is a circle of radius $\sqrt{5}$.

5. If the argument of the complex number $\frac{z-1}{z+1}$ is $\frac{\pi}{4}$, show that z lies on a fixed circle whose centre is at the point representing i.

6. Find the locus of z if **(i)** $|\frac{z-i}{z+2}| = 1$ **(ii)** $\arg(\frac{z-i}{z+2}) = \frac{\pi}{2}$ **(iii)** $|\frac{z-2}{z+2}| \le 1$ **(iv)** $2(z+\overline{z}) - 5i(z-\overline{z}) = 21$ **(v)** $z\overline{z} - (2+i)z - (2-i)\overline{z} \le 4$ **(vi)** $\arg(\frac{z-2}{z+2}) = 0$ **(vii)** $|z+3i|^2 + |z-3i|^2 = 90$ **(viii)** $|z - 3 - 2i| = |z - 2 + i|$.

*Other resources by James Coroneos are available. Write to P.O. Box 25, Rose Bay, NSW, 2029, Australia, for a catalogue. Typeset by $\mathcal{AMS}$-TeX.

7. If $z(w+1) = w-1$, show that as Z describes the y axis, W describes a circle with the origin as centre, and that, as Z describes the x axis, W describes the x axis also.

8. If $w = \frac{z+2i}{2iz-1}$, show that when z describes the circle $|z| = 1$ completely in one direction, then w describes the circle $|w| = 1$ completely in the other direction.

9. Two complex numbers z, Z are related by $Z = \frac{2+z}{2-z}$. Show that as the point z describes the y axis from the negative end to the positive end, the point Z describes completely the circle $x^2 + y^2 = 1$ in the counter-clockwise sense.

10. Given t is a real variable, find the locus of the point z on the Argand diagram such that **(i)** $z = \frac{2+it}{2-it}$ **(ii)** $z = 3 + \frac{4(1+it)}{1-it}$ **(iii)** $z = 2i + \frac{1+it}{1-it}$

11. If the point z moves on a semicircle, centre the origin and radius 2, in an anti-clockwise direction from the point 2 to the point -2, find the path traced by the point $1/z$.

12. If $\frac{z^2}{(z-1)}$ is always real, show that the locus of the point representing z consists of the real axis and a circle.

13. Prove that if z lies on the circle $x^2+y^2 = 1$, the points representing $Z = \sqrt{\frac{1+z}{1-z}}$ lie on an orthogonal line pair.

14. If P, Q represent the complex numbers z, Z and $Z = \frac{1}{z-3} + \frac{17}{3}$, find the locus of Q as P moves on the circle $|z-3| = 3$.

15. If θ is acute and $z_1 = \cos\theta + i\sin\theta, z_2 = \cos(\theta + \frac{2\pi}{3}) + i\sin(\theta + \frac{2\pi}{3})$, $z_3 = \cos(\theta + \frac{4\pi}{3}) + i\sin(\theta + \frac{4\pi}{3})$, show that
(i) z_1, z_2, z_3 are equally spaced around the unit circle.
(ii) $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1| = \sqrt{3}$ **(iii)** $\arg(z_1 - z_3) = \theta + \frac{\pi}{6}$

16. Obtain algebraically and geometrically, the complex numbers z which satisfy $|z| = 15$ and $|z-4| = 13$.

17. Determine the number z which satisfies the equations
(i) $|z+3i| = |z+5-2i|$ and $|z-4i| = |z+2i|$ simultaneously.
(ii) $|z-1| = 2\sqrt{2},\ |z-1-i| = |z|$ simultaneously.

18. If $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$, show algebraically and geometrically, that $|z_1 - z_2|^2 = r_1^2 + r_2^2 - r_1 r_2 \cos(\theta_1 - \theta_2)$.

19. Investigate the loci given by $|z|^2 = 2\Re(z)$ and $|z|^2 = 2\Im(z)$. Show that they meet at right angles at each common point.

20. O, A, B are the points $0, 1, 1+i$ respectively. If $w = z^2 + 1$, find the path traced out by w as z traces out the triangle OAB.

21. If $z = x+iy$, $w = u+iv$ are complex numbers and $w = \frac{2z}{1+|z|^2}$, find the curves in the z plane on which

- **(i)** the real part of w is constant
- **(ii)** the imaginary part of w is constant.

22. If z is a complex number whose imaginary part is non-zero, and $z + 1/z$ is real, what is $|z|$?

23. z, w are complex numbers. Prove that if $\frac{i(w+z)}{w-z}$ is real, then $|z| = |w|$.

24. If $z = \cos\theta + i\sin\theta$ and $w = \frac{1-z}{1+z}$, show that the point representing w on the Argand diagram lies on the imaginary axis and find its coordinates in terms of $\theta/2$.

25. Prove that $\frac{|z|-iz}{|z|+iz} = -i(\sec\theta + \tan\theta)$, where $\Re(z) \neq 0$ and $\arg z = \theta$.

26. Two complex variables z and w are connected by the relation $zw+z-w+1 = 0$. If $z = 2 + e^{i\theta}$, prove that $w = -2 + i\tan\frac{\theta}{2}$, and describe the loci of the points z and w as θ varies from $-\pi$ to $+\pi$.

27. If z is a complex number for which $|z| = 1$ and $\arg z = \theta$, find the values of $|\frac{2}{1-z^2}|$ and $\arg(\frac{2}{1-z^2})$. $\{$Assume $0 < \theta < \frac{\pi}{2}\}$

28. On an Argand diagram, show the points P, Q corresponding to the complex numbers $p = z_1 + z_2, q = z_1 - z_2$, where z_1, z_2 are given complex numbers. Show that if O denotes the point corresponding to zero, and

- **(i)** if $P\hat{O}Q = \frac{\pi}{2}$, then $|z_1| = |z_2|$
- **(ii)** if $|OP| = |OQ|$, then $\frac{z_2^2}{z_1^2}$ is real and negative.

29. If ρ is real, and the complex number $\frac{1+i}{2+\rho i} + \frac{2+3i}{3+i}$ is represented in an Argand diagram by a point on the line $x = y$, show that $\rho = -5 \pm \sqrt{21}$.

30. If $z = 1 + \cos 2\theta + i\sin 2\theta$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, prove that $|z| = 2\cos\theta$ and $\arg z = \theta$. Describe the path traced out by the point z as θ varies from $-\pi/2$ to $+\pi/2$.

31. If A, B, P, Q are the four points $(-1, 0), (1, 0), (x, y), (u, v)$ respectivley; if x, y, u, v are real and $u+iv = \frac{1}{2}[x+iy+(x+iy)^{-1}]$, prove that $|AQ| : |BQ| = AP^2 : BP^2$