NEW SOUTH WALES

Higher School Certificate

Mathematics Extension 2

Exercise 8/67

by James Coroneos*

1. If $(1-i)z = 3-4i$, find
(i) z **(ii)** $|z|$ **(iii)** $\overline{z}$ **(iv)** $z-\overline{z}$ **(v)** $\Im(z)$ **(vi)** $\Re(z)$ **(vii)** $z\overline{z}$ **(viii)** $|\frac{\overline{z}}{z}|$
(ix) $\arg(\overline{z})$

2. Given that $z_1 = 2-i$, $z_2 = 1+2i$, $z_3 = i-1$, determine $|z_1z_2z_3|$ and $|\frac{z_1z_2}{z_3}|$. On the Argand diagram, mark the points $z_1, z_2, z_3, z_1z_2, \frac{z_1z_2}{z_3}, z_1z_2z_3$.

3. Draw a diagram to represent the complex numbers $z_1 = \sqrt{3}+i$, $z_2 = \sqrt{2}+i\sqrt{2}$. What are the moduli and arguments of these numbers? Illustrate on the same diagram, the numbers z_1z_2 and z_1/z_2, and find their moduli and arguments. Hence show that $|z_1z_2| = |z_1||z_2|$ and $\arg(z_1z_2) = \arg z_1 + \arg z_2$. Similarly, prove that $|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|}$ and $\arg(\frac{z_1}{z_2}) = \arg z_1 - \arg z_2$.

4. By writing $15° = 45° - 30°$, show that $\cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$, and find $\sin 15°$. Given that $z = \frac{1+\sqrt{3}i}{1+i}$, show by the two methods illustrated in question **3** above, that $|z| = \sqrt{2}$ and $\arg z = \frac{\pi}{12}$.

5. If $z = (1+\sqrt{3}i)(1+i)$, show that $z = (1-\sqrt{3}) + (1+\sqrt{3})i$. Also prove that $|z| = 2\sqrt{2}$ and $\arg z = \frac{\pi}{3} + \frac{\pi}{4}$. *Hence* deduce that $\cos 105° = \frac{1-\sqrt{3}}{2\sqrt{2}}$ and $\sin 105° = \frac{1+\sqrt{3}}{2\sqrt{2}}$. By using the $\cos 2A, \sin 2A$ results, find $\cos 210°$ and $\sin 210°$.

*Other resources by James Coroneos are available. Write to P.O. Box 25, Rose Bay, NSW, 2029, Australia, for a catalogue. Typeset by $\mathcal{AMS}$-TEX.

6. If two complex numbers are equal, then their moduli and arguments are equal; and convesely, if two complex numbers have their moduli and arguments equal, then they themselves are equal. Furthermore, $\left|\frac{\prod_{j=1}^{n} z_j}{\prod_{k=1}^{m} z_k}\right| = \frac{\prod_{j=1}^{n} |z_j|}{\prod_{k=1}^{m} |z_k|}$ and $\arg\left(\frac{\prod_{j=1}^{n} z_j}{\prod_{k=1}^{m} z_k}\right) = \sum_{j=1}^{n} \arg z_j - \sum_{k=1}^{m} \arg z_k$ (which shows that the argument function is actually a logarithm). Hence $|z^n| = |z|^n$ and $\arg(z^n) = n \arg z$. Using these theorems, where $z = r(\cos\theta + i\sin\theta)$, prove (where the arguments are not necessarily principal) that

(i) $|(1-i)z^2| = \sqrt{2}r^2$ **(ii)** $\arg\{(1-i)z^2\} = 2\theta - \frac{\pi}{4}$ **(iii)** $|iz^3| = r^3$

(iv) $\arg(iz^3) = \frac{\pi}{2} + 3\theta$ **(v)** $|\frac{1+i\sqrt{3}}{z}| = \frac{2}{r}$ **(vi)** $\arg(\frac{1+i\sqrt{3}}{z}) = \frac{\pi}{3} - \theta$

(vii) $\left|\frac{3(2+2i)}{(\sqrt{3}-i)z^2}\right| = \frac{3\sqrt{2}}{r^2}$ **(viii)** $\arg\left(\frac{3(2+2i)}{(\sqrt{3}-i)z^2}\right) = \frac{5\pi}{12} - 2\theta$

(ix) $\arg(-\frac{1}{z^3}) = \pi - 3\theta$ **(x)** $|z(\frac{3}{4} - i)(1+i)| = \frac{5\sqrt{2}r}{4}$

(xi) $\arg\{i(i-1)(i-\sqrt{3})z\} = \frac{25\pi}{12} + \theta$ **(xii)** $|-\frac{1}{z^3}| = \frac{1}{r^3}$

7. Find the modulus and argument of the following, and express each in mod-arg form. **(i)** $(\sqrt{3}+i)(1-i)$ **(ii)** $(-\sqrt{3}+i)(1-i)$ **(iii)** $\frac{(i-1)}{\sqrt{3}-i}$ **(iv)** $-\frac{2}{1+i}$
(v) $(2+3i)(1-i)$ **(vi)** $\frac{2+3i}{1-i}$ **(vii)** $\frac{2+3i}{3+2i}$ **(viii)** $(1-i)^{-2}$ **(ix)** $\frac{3(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})}{2(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})}$

8. Find the real part, imaginary part, modulus, argument and complex conjugate of each of the following $(0 < \theta < \frac{\pi}{2})$:
(i) $\frac{1}{1+i}$ **(ii)** $\frac{1-\cos 2\theta + i\sin 2\theta}{1+\cos 2\theta - i\sin 2\theta}$ **(iii)** $\frac{1}{4-3i} - \frac{i}{3-4i}$

9. **(i)** If $z = \frac{7-i}{3-4i}$, find $\Re(z)$, $\Im(z)$, $|z|$, $\arg z$, $\overline{z}$
(ii) If $\frac{1}{z} = \frac{1}{i\sqrt{3}-3} + \frac{2}{i\sqrt{3}+3}$, express z in mod-arg form.

10. Find $|z|$ and $\arg z$ if $\frac{3z+2}{4z-1} = i$

11. **(i)** If $(x+iy)^n = a+ib$, find a^2+b^2.
(ii) If $A+iB = (a_1+ib_1)(a_2+ib_2)$, prove that $(a_1^2+b_1^2)(a_2^2+b_2^2) = A^2+B^2$.

12. Evaluate $f(3-5i)$, given that **(i)** $f(z) = z^2 - 3z + 2$ **(ii)** $f(z) = |\frac{z^2-1}{\overline{z}+1}|$

13. Show that
$(\cos 230^\circ + i\sin 230^\circ)(\cos\theta + i\sin\theta)^2 = \cos(2\theta + 230^\circ) + i\sin(2\theta + 230^\circ)$.

Hence, if
$$z_1 = 12(\cos 130^\circ + i\sin 130^\circ),$$
$$z_2 = 15(\cos 20^\circ - i\sin 20^\circ),$$
$$z_3 = 20(\cos 230^\circ + i\sin 230^\circ),$$
$$z_4 = r(\cos\theta + i\sin\theta) \text{ and } z_3 z_4^2 = z_1 z_2,$$
find r and θ, given $r > 0$ and $-\frac{\pi}{2} < \theta < 0$.

14. **(i)** Assuming $|z_1 z_2| = |z_1| \cdot |z_2|$, prove by induction that $|z_1 z_2 z_3 \cdots z_2| = |z_1||z_2||z_3| \cdots |z_n|$.

(ii) Assuming $\arg z_1 z_2 = \arg z_1 + \arg z_2$, prove by induction that $\arg(z_1 z_2 z_3 \cdots z_n) = \arg z_1 + \arg z_2 + \cdots + \arg z_n$