NEWCASTLE GRAMMAR SCHOOL

Student Number: ______________________________

2011
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Examination Date: Friday 19th August

Examiner: Mr. M. Brain

General Instructions

- Reading Time - 5 minutes
- Working Time - 2 hours
- Write using a blue or black pen
- Write your student number on every booklet
- Board-approved calculators may be used
- A table of standard integrals is provided in this paper
- All necessary working should be shown in every question
- Each question attempted is to be returned in a separate Writing Booklet clearly marked Questions 1 etc.
- If required, additional booklets may be requested

Total marks - 84

- Attempt Questions 1- 7
- All questions are of equal value

Question 1 (Start a new booklet) **Marks**

a) If $y = \left(\tan^{-1} x\right)^2$ find $\frac{dy}{dx}$ **2**

b) Find the value of $\sum_{n=2}^{5} {}^{n}C_2$ **2**

c) Solve $\frac{2x-3}{x-2} \geq 1$ **4**

d) Find $\lim_{x \to 0} \frac{\sin 3x}{x}$ **1**

e) Evaluate $\int_0^3 \frac{x}{\sqrt{1+x}}\,dx$ using the substitution $x = u^2 - 1$, where $u > 0$ **3**

Question 2 (Start a new booklet) **Marks**

a) Find the size of the acute angle between the two lines with equations $x+2y+1=0$ and $2x-3y+6=0$, correct to the nearest degree **4**

b) A spherical balloon is expanding so that its volume, V cm^3, increases at a constant rate of 72 cm^3 per second. What is the rate of increase of the surface area when the radius is 12 cm? **4**

c) Find **4**

(i) $\int \sin^2 x\, dx$

(ii) $\int \frac{dx}{\sqrt{9+4x^2}}$

Question 3 (Start a new booklet) **Marks**

a) The rate of growth of a bacteria colony is proportional to the excess of the colony's population over 5000 and is given by **5**

$$\frac{dN}{dt} = k(N - 5000)$$

(i) Show that $N = 5000 + Ae^{kt}$ is a solution to the differential equation above

(ii) Given that the initial population was 15000 and had risen to 20000 after 2 days find the value of A and k

(iii) Hence calculate the expected population after 7 days

b) The polynomial $P(x) = x^5 + mx^3 + nx$ has a remainder of 5 when divided by $(x-2)$, where m and n are constants. **3**

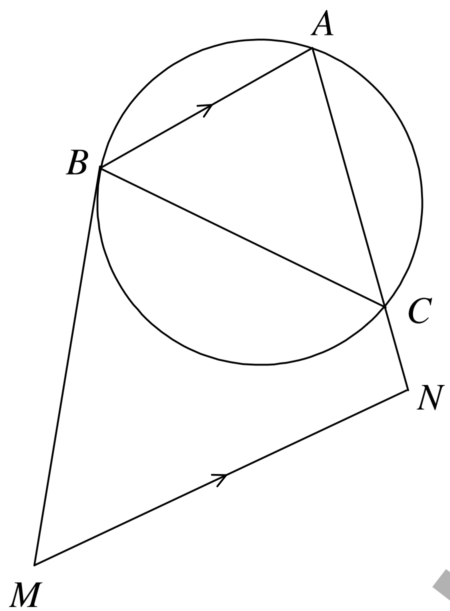
(i) Prove that $P(x)$ is an odd function

(ii) Hence find the remainder when $P(x)$ is divided by $(x+2)$

c) If α, β and γ are the roots of the equation $x^3 - 2x^2 + 3x + 7 = 0$ find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **4**

Question 4 (Start a new booklet) **Marks**

a) ABC is a triangle inscribed in a circle. M is a point on the tangent to the circle at B and N is a point on AC produced so that MN is parallel to BA **4**



Copy the diagram into your answer booklet

(i) State why $\angle MBC = \angle BAC$

(ii) Prove that $MNCB$ is a cyclic quadrilateral

b) Prove by mathematical induction that, for all integers $n \geq 1$ **4**

$$1\times 2^0 + 2\times 2^1 + 3\times 2^2 + \ldots + n\times 2^{n-1} = 1 + (n-1)2^n$$

c) (i) Prove that the equation $\dfrac{\log_e x}{x} + 2 = 0$ has a solution between $x = 0\cdot 4$ and $x = 0\cdot 5$ **4**

(ii) Use one application of Newton's method to find a closer approximation to the solution $x = 0\cdot 4$, correct to three decimal places

Question 5 (Start a new booklet) **Marks**

a) In Group A there are 5 men and 3 women. In Group B there are 4 men and 6 women. **4**

(i) If one person is chosen at random from each group what is the probability that the two people chosen are of opposite sexes?

(ii) If a group and then one person from that group is chosen at random what is the probability that the person chosen is a man?

b) Using $\tan\frac{\theta}{2} = t$ show that $\frac{1-\cos\theta}{\sin\theta} = t$ **3**

c) A particle moves in a straight line such that its acceleration, a, is given by $a = 3x^2$, where x is displacement, v is velocity and t is time. Given that $v = -\sqrt{2}$ and $x = 1$ when $t = 0$ find x as a function of time, t **5**

Question 6 (Start a new booklet) **Marks**

a) Consider the function $f(x) = 3\sin^{-1}\left(\frac{x}{2}\right)$ **5**

(i) State the domain and range of this function

(ii) Sketch the graph of $y = f(x)$

(iii) Find the slope of the graph at $x = 0$

b) The displacement, x metres, of a particle, at t seconds is given by: **7**

$$x = 5\cos(4\pi t)$$

(i) Show that the acceleration of the particle can be expressed in the form:

$$\ddot{x} = -n^2 x$$

(ii) State the period, P, of the motion

(iii) Determine the maximum velocity of the particle

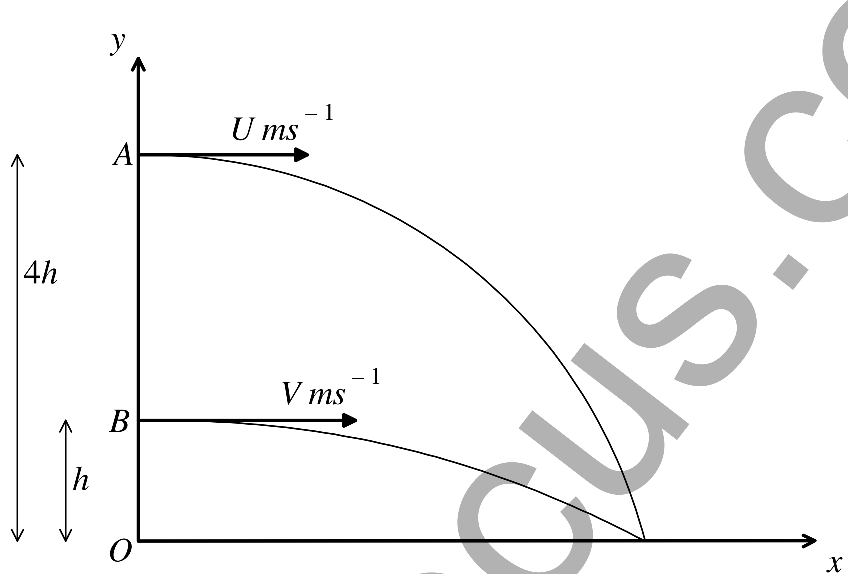
(iv) Determine the maximum acceleration of the particle

(v) Express v^2 in terms of x, where v is the velocity of the particle

<u>QUESTION 7</u> IS ON THE NEXT PAGE

Question 7 (Start a new booklet) **Marks**

a) A vertical building stands with its base O on horizontal ground. A and B are two points on the building vertically above each other such that A is $4h$ metres above O and B is h metres above O. A particle is projected horizontally with speed U metres per second from A and 10 seconds later a second particle is projected horizontally with speed V metres per second from B. The two particles hit the ground at the same point and at the same time. **7**



(i) Show that the horizontal and vertical displacements of particles A and B, t seconds after the first particle is projected, are given by $x_A = Ut$, $y_A = 4h - \frac{1}{2}gt^2$ and $x_B = V(t-10)$, $y_B = h - \frac{1}{2}g(t-10)^2$, respectively

(ii) Find the time of flight of each particle

(iii) Prove that $V = 2U$

b) The point $P(2t, t^2)$ is on the parabola with equation $x^2 = 4y$, having its focus at F. The point M divides the interval FP externally in the ratio 3:1. **5**

(i) Show that the co-ordinates of M are $x = 3t$ and $y = \frac{1}{2}(3t^2 - 1)$

(ii) Hence prove that the locus of M is also a parabola and determine the focal length of the locus of M

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ \ n \neq -1;\ \ x \neq 0,\ \text{if } n<0$$

$$\int \frac{1}{x}dx = \ln x,\ \ x>0$$

$$\int e^{ax}dx = \frac{1}{a}e^{ax},\ \ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ \ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a},\ \ a>0,\ \ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ \ x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\ \ x>0$