Normanhurst Boys High School

2013 HSC TRIAL EXAMINATION

MATHEMATICS EXTENSION 1 – MARKING GUIDELINES

Section I

Question	Marks	Answer	Outcomes Assessed
1	1	~~B~~C	O1
2	1	A	O1
3	1	A	O1
4	1	D	O3
5	1	A	O4
6	1	C	O4
7	1	C	O4
8	1	B	O5
9	1	C	O5
10	1	C	O5

Question 11 (15 marks)

11(a) (2 marks)
Outcomes Assessed: O3

Criteria	Marks
• Use sine of the sum	1
• Correct answer	1

Answer

$$\sin 75^\circ = \sin(45^\circ + 30^\circ)$$
$$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$
$$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2}$$
$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

11(b) (3 marks)
Outcomes Assessed: O3

Criteria	Marks
• Obtains correct gradients	1
• Correct substitution into formulae	1
• Correct answer	1

Answer

$$\tan 45^\circ = \left|\frac{m - \frac{2}{3}}{1 + \frac{2m}{3}}\right|$$
$$\left|\frac{3m-2}{3} \div \frac{3+2m}{3}\right| = 1$$
$$\left|\frac{3m-2}{3+2m}\right| = 1$$
$$\frac{3m-2}{3+2m} = 1 \quad or \quad \frac{3m-2}{3+2m} = -1$$
$$m = 5 \quad or \quad m = -\frac{1}{5}$$

11(c) (3 marks)
Outcomes Assessed: O5

Criteria	Marks
• Obtains correct limits	1
• Obtains $I = \frac{1}{2}\int_1^9 \left(\frac{du}{u^{\frac{1}{2}}}\right)$	1
• Correct answer	1

Answer

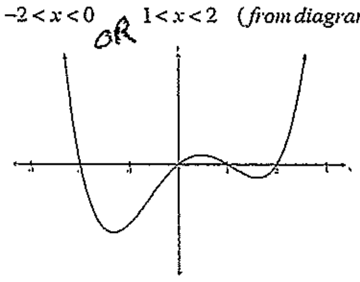
$$u = 1 + x^2$$
$$\frac{1}{2}du = x\,dx$$
$$x = \sqrt{8} \to u = 9$$
$$x = 0 \to u = 1$$
$$I = \int_0^{\sqrt{8}} \left(\frac{x}{\sqrt{1+x^2}}\right) dx$$
$$I = \frac{1}{2}\int_1^9 \left(\frac{du}{u^{\frac{1}{2}}}\right)$$
$$= \left[u^{\frac{1}{2}}\right]_1^9$$
$$= 3 - 1$$
$$= 2$$

11(d) (3 marks)
Outcomes Assessed: O2

Criteria	Marks
• Multiplies throughout by $x^2(x-2)^2$	1
• Obtain $x(x-2)(x+2)(x-1) < 0$	1
• Correct answer	1

Answer

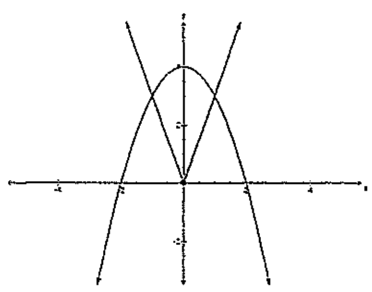
$$\frac{1}{x} + \frac{x}{(x-2)} < 0$$
$$x(x-2)^2 + x^3(x-2) < 0$$
$$x(x-2)\left[(x-2) + x^2\right] < 0$$
$$x(x-2)\left[x^2 + x - 2\right] < 0$$
$$x(x-2)(x+2)(x-1) < 0$$
$-2 < x < 0$ OR $1 < x < 2$ *(from diagram)*



11(e) (i) (2 marks)
Outcomes Assessed: O1

Criteria	Marks
• Correct graph for $y = 4 - x^2$	1
• Correct graph for $y = \lvert 3x \rvert$	1

Answer



11(e) (ii) (2 marks)
Outcomes Assessed: O1

Criteria	Mark
• Solve for both points of intersection	1
• Correct answer	1

Answer

Solve
$4-x^2=|3x|$
$4-x^2=3x \quad or \quad 4-x^2=-3x$
$x^2+3x-4=0 \quad or \quad x^2-3x-4=0$
$(x+4)(x-1)=0 \quad or \quad (x-4)(x+1)=0$
$x=1,-4 \qquad x=4,-1$
check solutions
correct solutions : $x=\pm 1$
hence from diagram $x<-1$ *or* $x>1$

Question 12 (15 marks)
12(a) (2 marks)
Outcomes Assessed: O4

Criteria	Marks
• Achieves $\cos\alpha=\frac{\sqrt{3}}{4}$ and $\sin\alpha=\frac{\sqrt{13}}{4}$	1
• Correct answer	1

Answer

$Let\ \alpha=\cos^{-1}\frac{\sqrt{3}}{4}$

$\therefore \sin\left(2\cos^{-1}\frac{\sqrt{3}}{4}\right)=\sin(2\alpha)$

Also $\cos\alpha=\frac{\sqrt{3}}{4}$, *hence* $\sin\alpha=\frac{\sqrt{13}}{4}$ (*pythagoras*)

$$\sin\left(2\cos^{-1}\frac{\sqrt{3}}{4}\right)=\sin(2\alpha)$$
$$=2\sin\alpha\cos\alpha$$
$$=2.\frac{\sqrt{13}}{4}.\frac{\sqrt{3}}{4} \quad = \quad \frac{\sqrt{39}}{8}$$
$$=\cancel{\frac{\sqrt{69}}{8}}$$

12(b) (i) (1 mark)
Outcomes Assessed: O3

Criteria	Mark
• Correct answer reasoning	1

Answer

$AO=OB$ (radii of circle centre O)

12(b) (ii) (2 marks)
Outcomes Assessed: O3

Criteria	Marks
• Notes that *Also* $\angle AZO=90°$ with reasons	1
• Correct answer with correct reasoning	1

Answer

If from (i) ΔOAB *is isosceles*
Also $\angle AZO=90°$ (*angles in a semi-circle are right angles at the circumference*)
$\therefore AZ=ZB$ (*a line from the apex of an isosceles triangle which meets the base at right angles, bisects the base*)

12(c) (3 marks)
Outcomes Assessed: O3

Criteria	Marks
• Obtains $\tan A+\tan B=4$ or $\tan A\tan B=9$	1
• Uses $\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$	1
• Correct answer	1

Answer

$\tan A+\tan B=-\frac{b}{a}$
$\tan A+\tan B=4$
$\tan A\tan B=\frac{c}{a}$
$\tan A\tan B=9$
$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$
$$=\frac{4}{1-9}$$
$$=-\frac{1}{2}$$
$$\angle(A+B)=153°26',\ 333°26'$$
$$=153°,\ 333°$$

12(d) (i) (2 marks)
Outcomes Assessed: O4

Criteria	Marks
• Makes a positive attempt to solve obtain the remainder by long division	1
• Correct answer	1

Answer
$P(x)=x^4+3x^3+6x^2+ax+b$
By long division
$P(x)=(x^2+2x+1)(x^2+x+3)+[(a-7)x+(b-3)]$
$\therefore R(x)=(a-7)x+(b-3)$

12(d) (ii) (1 mark)
Outcomes Assessed: O4

Criteria	Mark
• Correct answer	1

Answer
$3x+2=(a-7)x+(b-3)$
$\therefore a=10$ and $b=5$

12(e) (i) (2 marks)
Outcomes Assessed: O3

Criteria	Marks
• Uses the double angle correctly at least once	1
• Correct answer with correct working	1

Answer
$\sin A\cos A\cos 2A=\frac{1}{4}\sin 4A$

$$LHS=\frac{1}{2}[2\sin A\cos A\times\cos 2A]$$
$$=\frac{1}{2}[\sin 2A\times\cos 2A]$$
$$=\frac{1}{2}\times\frac{1}{2}[2\sin 2A\times\cos 2A]$$
$$=\frac{1}{4}\sin 4A$$
$$=RHS$$

12(e) (ii) (2 marks)
Outcomes Assessed: O3

Criteria	Marks
• Achieves at least 2 of the correct answers	1
• Correct answer	1

Answer
$\frac{1}{4}\sin 4A=0$
$\sin 4A=0$
$4A=0,\pi,2\pi,3\pi,4\pi.....$
$A=0,\frac{\pi}{4},\frac{\pi}{2},\frac{3\pi}{4},.....$

Hence $A=0,\frac{\pi}{4},\frac{\pi}{2} \quad ;0\le A\le\frac{\pi}{2}$

Question 13 (15 marks)
13(a) (3 marks)
Outcomes Assessed: O2

Criteria	Marks
• ~~Provides clear~~ steps similar to the first three steps below	1
• Makes a substitution in step 3 ~~(from step 2)~~	1
• Provides a clear working ~~to obtain the required result and provides a conclusion~~	1

Answer

$5^n\ge 1+4n$

Step 1&3
Step 2 – Using assumption for n=k
– All correct

Step 1: Prove the expression is true for n=1
$5\ge 5$ (*true*) for n=1
$LHS=5^1=5$
$RHS=1+4\times1=5$

Assume the expression is true for n=k (where k is even)
i.e. $5^k\ge 1+4k$ (where k is a positive integer)

Step 2 : Prove the expression is true for n=k+1
i.e. $5^{k+1}\ge 1+4(k+1)$
$5.5^k\ge 4k+5$

$5^{k+1}-4k-5\ge 0$
Now $LHS=5.5^k-4k-5$
$\ge 5(1+4k)-4k-5$ (from assumption)
$=16k\ge 0 \qquad (k\ge 1)$

Hence if the expression is true when n=k, it is true when n=k+1
Step 3 If the expression is true for n=1, $\therefore$ it is true when n=2
If true for n=2, $\therefore$ it is true when n=3
Therefore the expression is true for all n, $n\ge 1$.

3

13(b) (i) (2 marks)
Outcomes Assessed: O4

Criteria	Marks
• Show gradient of tangent at P = p	1
• Achieves $y = px - ap^2$ with sufficient working	1

Answer $x^2 = 4ay$

$$\frac{dy}{dx} = \frac{x}{2a} = \frac{2ap}{2a} = p$$

Gradient of tangent at $P = \frac{\cancel{2}ap}{\cancel{2}a} = p$

Equation of tangent at P
$$y - ap^2 = p(x - 2ap)$$
$$y - ap^2 = px - 2ap^2$$
$$y = px - ap^2$$

2

13(b) (ii) (1 mark)
Outcomes Assessed: O4

Criteria	Marks
• Correct working to achieve Z	1

Answer

Tangent: $y = px - ap^2$
For Z: substitute $y=0$
$px - ap^2 = 0$
$x = ap$
$Z = (ap, 0)$

1

13(b) (ii) (2 marks)
Outcomes Assessed: O4

Criteria	Marks
• Obtains midpoint$_{ZQ} = \left(\frac{-ap}{2}, \frac{ap^2}{2}\right)$	1
• Correct answer for locus	1

Answer

$Q = (-2ap, ap^2)$ symmetry of parabola

$\therefore$ midpoint$_{ZQ} = \left(\frac{-ap}{2}, \frac{ap^2}{2}\right)$ $\left(\frac{-2ap+ap}{2}, \frac{ap^2+0}{2}\right)$

hence $x = \frac{-ap}{2}$ — 1

$p = \frac{-2x}{a}$

Sub into 2,

$y = \frac{ap^2}{2}$ — 2

$= \frac{a}{2}\left(\frac{-2x}{a}\right)^2$

$y = \frac{2x^2}{a}$

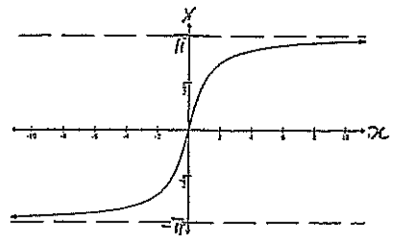
$\therefore 2x^2 = ay$ is the locus of midpoints of QZ.

2

13(c) (i) (1 mark)
Outcomes Assessed: O4

Criteria	Mark
• Correct ~~diagram~~ graph with asymptotes labeled.	1

Answer



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13(c) (ii) (1 mark)
Outcomes Assessed: O4

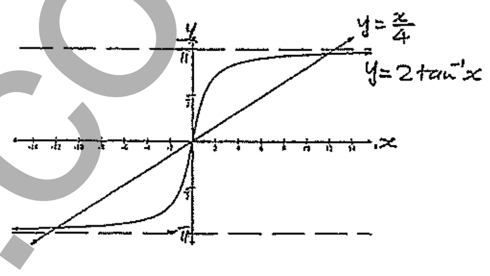
Criteria	Mark
• Correct answer	1

Answer

$2\tan^{-1}(x) - \frac{x}{4} = 0$

$2\tan^{-1}(x) = \frac{x}{4}$

$\therefore$ Graph $y = 2\tan^{-1}(x)$ and $y = \frac{x}{4}$ to show that there is only one point of intersection (12, 3) for $x > 0$



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13(c) (iii) (2 marks)
Outcomes Assessed: O4

Criteria	Marks
• ~~Uses the correct formulae and makes a good attempt at achieving answer~~	~~1~~
• ~~Correct answer~~	~~1~~

Answer

$P(x) = 2\tan^{-1}(x) - \frac{x}{4}$

$P'(x) = \frac{2}{1+x^2} - \frac{1}{4}$

$x_2 = x_1 - \frac{P(x_1)}{P'(x_1)}$

$= 10 - \frac{P(10)}{P'(10)}$

$= 10 + \frac{0.4423}{0.230}$

$= 11.92$ (2 *decimal places*)

2

13(d) (3 marks)
Outcomes Assessed: O5

Criteria	Marks
• Differentiates correctly	1
• ~~Equates various rates correctly~~	1
• ~~Achieves correct answer~~	1

Answer

$x^2 + y^2 = 25$

$y = \sqrt{25 - x^2} \quad (y > 0)$

$\frac{dy}{dx} = \frac{-x}{\sqrt{25 - x^2}}$

Now $\frac{dx}{dt} = 1$

$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

$\frac{dy}{dt} = \frac{-x}{\sqrt{25 - x^2}} \times 1$

Now when $y = 4$, $x = 3$

$\frac{dy}{dt} = \frac{-3}{\sqrt{25 - 3^2}} \times 1$

$= -\frac{3}{4}$ (*i.e.* $\frac{3}{4}$ *metres per second down the wall*)

3

Note: A quicker method using the implicit function rule can be used by Ext 2 students.

Question 14 (15 marks)
14(a) (i) (2 marks)
Outcomes Assessed: O5

Criteria	Marks
• ~~Achieves the acceleration formulae in any form~~	1
• ~~Correct working~~	1

Answer

$x = 2 + \sin^2 t$ — 1

$\dot{x} = v = 2\sin t\cos t$

$\ddot{x} = \cos t(2\cos t) + 2\sin t(-\sin t)$

$= 2[\cos^2 t - \sin^2 t]$

$= 2[1 - \sin^2 t - \sin^2 t]$

$= 2[1 - 2\sin^2 t]$

$= 2[1 - 2(x-2)]$ using 1

$= 2[1 - 2x + 4]$

$= 10 - 4x$

$= -4\left(x - \frac{5}{2}\right)$

Alternative solution

$\dot{x} = \sin 2t$

$\ddot{x} = 2\cos 2t$

$= 2(1 - 2\sin^2 t)$

$= 2[1 - 2(x-2)]$ using 1

$= -4(x - \frac{5}{2})$

Since in the form of $\ddot{x} = -n^2 X$, therefore SHM.
where $X = (x - \frac{5}{2})$ and $n = 2$

2

14(a) (ii) (1 mark)
Outcomes Assessed: O5

Criteria	Mark
• Correct answer	1

Answer

$\ddot{x} = 0$

$10 - 4x = 0$

$x = \frac{5}{2}$

1

14(a) (iii) (2 marks)
Outcomes Assessed: O5

Criteria	Marks
• Achieves $t=0,\frac{\pi}{2},\pi,\frac{3\pi}{2}.....$	1
• Correct answer with sufficient working	1

Answer

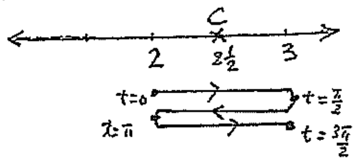
$v=0$ ← Particle changes direction

$v=2\sin t\cos t$

$2\sin t\cos t=0$

$\therefore \sin t=0 \qquad \cos t=0$

$t=0,\pi,2\pi,3\pi... \qquad t=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2}$



$t=0,\frac{\pi}{2},\pi,\frac{3\pi}{2}.....$ ✓

when $t=0 \quad x=2$

$t=\frac{\pi}{2} \quad x=3$

$t=\pi \quad x=2$

$t=\frac{3\pi}{2} \quad x=3$ ✓

$\therefore$ total distance $=3\,cm$

②

14(b) (3 marks)
Outcomes Assessed: O3

Criteria	Mark
• Find AC	1
• Find $\angle ABC$	1
• Correct answer to area	1

Answer

$\Delta ABD, \quad AB^2=2.4^2+6.4^2$
$\qquad AB\approx 6.835m$

$\Delta CBD, \quad BC^2=2.4^2+5.2^2$
$\qquad BC\approx 5.727m$

$\Delta ACD, \quad AC^2=6.4^2+5.2^2-2\times 6.4\times 5.2\times cos125°$ ✓
$\qquad AC\approx 10.304m$

$\Delta ABC, \quad cos\angle ABC=\frac{AB^2+BC^2-AC^2}{2(AB)(BC)}$
$\qquad \approx \frac{46.72+32.8-106.17}{2(6.835)(5.727)}$

$\angle ABC\approx 109.9°$ ✓

Area $\Delta ABC=\frac{1}{2}\times 6.835\times 5.727\times sin109.9°$
$\qquad =18.4\,m^2$ (1dp) ✓

③

14(c) (i) (2 marks)
Outcomes Assessed: O5

Criteria	Marks
• Show $\dot{x}$ and $\dot{y}$ with sufficient working	1
• Correct x and y with working shown	1

Answer

$\tan\alpha=\frac{5}{12}, \therefore \sin\alpha=\frac{5}{13}$ and $\cos\alpha=\frac{12}{13}$

$\ddot{x}=0 \qquad \ddot{y}=-10$

$\dot{x}=v\cos\alpha \qquad \dot{y}=-10t+v\sin\alpha$

$\dot{x}=130\times\frac{12}{13} \qquad \dot{y}=-10t+130\times\frac{5}{13}$

$\dot{x}=120 \qquad \dot{y}=-10t+50$

$x=120t \qquad y=-5t^2+50t+15$

$\ddot{x}=0$
$\dot{x}=C_1$
when $t=0$, $\dot{x}=v\cos\alpha$
$\therefore C_1=v\cos\alpha$
$\dot{x}=v\cos\alpha$
$=130\times\frac{12}{13}$
$=120$

$x=120t+C_3$
when $t=0$, $x=0$
$\therefore C_3=0$
$x=120t$

$\ddot{y}=-10$
$\dot{y}=-10t+C_2$
when $t=0$, $\dot{y}=v\sin\alpha$
$\therefore C_2=v\sin\alpha$
$\dot{y}=-10t+v\sin\alpha$
$=-10t+130\times\frac{5}{13}$
$=-10t+50$

$y=\frac{-10t^2}{2}+50t$
when $t=0$, $y=15$
$\therefore 15=C_4$
$y=-5t^2+50t+15$

14(c) (ii) (2 marks)
Outcomes Assessed: O5

Criteria	Marks
• Uses $v^2=\left(\dot{x}\right)^2+\left(\dot{y}\right)^2$ and works towards answer	1
• Correct answer	1

Answer

$v^2=\left(\dot{x}\right)^2+\left(\dot{y}\right)^2$

$\left(60\sqrt{5}\right)^2=(120)^2+(-10t+50)^2$ ✓

$18000=14400+100t^2-1000t+2500$

$100t^2-1000t-1100=0$

$t^2-10t-11=0$

$(t-11)(t+1)=0$

$t=11,-1$

$t=11 \quad (t>0)$

$x=120(11)$
$=1320m$ ✓

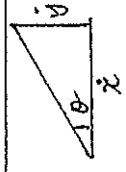
②

14(c) (iii) (3 marks)
Outcomes Assessed: O5

Criteria	Marks
• Uses $\tan\left(20^o\right)\le\left\|\frac{\dot{y}}{\dot{x}}\right\|\le\tan\left(30^o\right)$ or similar	1
• Flight path is in a downward direction $\therefore$ negative	1
• Correct answer	1

Answer

$20^\circ<\theta<30^\circ$



$\tan\theta=\frac{\dot{y}}{\dot{x}}$

$\tan\left(20^o\right)\le\left|\frac{\dot{y}}{\dot{x}}\right|<\tan\left(30^o\right) \rightarrow \tan 20°<\left|\frac{-10t+50}{120}\right|<\tan 30°$ ✓

$0.364<\left|\frac{-10t+50}{120}\right|<0.577$

$43.68<\left|-10t+50\right|<69.28$

As the flight path is in a downward direction $\dot{y}<0$. $\Longrightarrow 43.68<10t-50<69.28$

Hence

$50-10t<-43.68 \qquad 50-10t>-69.28$

$9.868\le t \qquad 11.928\ge t$ ✓

✓ $9.868<t<11.928$

$9.87<t<11.93$

③

END OF PAPER