HORNSBY GIRLS HIGH SCHOOL

Mathematics Extension 1

Year 12 Higher School Certificate
Trial Examination Term 3 2013

STUDENT NUMBER: _____

General Instructions

- Reading Time – 5 minutes
- Working Time – 2 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided seperately
- In Questions 11 – 14, show relevant mathematical reasoning and/or calculations
- Marks may be deducted for untidy and poorly arranged work
- Do not use correction fluid or tape
- Do not remove this paper from the examination

Total marks – 70

Section I Pages 3 – 6

10 marks

Attempt Questions 1 – 10

Answer on the Objective Response Answer Sheet provided

Section II Pages 7 – 11

60 marks

Attempt Questions 11 – 14.

Start each question in a new writing booklet.

Write your student number on every writing booklet.

Question	*1-10*	*11*	*12*	*13*	*14*	*Total*
Total	*/10*	*/15*	*/15*	*/15*	*/15*	*/70*

This assessment task constitutes 45% of the Higher School Certificate Course School Assessment

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Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the Objective Response answer sheet for Questions 1 – 10

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1 A polynomial equation has roots α, β and γ, where:
$\alpha+\beta+\gamma=-3$, $\alpha\beta+\alpha\gamma+\beta\gamma=-2$ and $\alpha\beta\gamma=4$.
Which polynomial equation has the roots α, β and γ?

(A) $x^3-3x^2-2x+4=0$

(B) $x^3+3x^2-2x-4=0$

(C) $x^3+3x^2+2x+4=0$

(D) $x^3-3x^2+2x-4=0$

2 The solution to $\dfrac{4}{x-3}\leq 2$ is:

(A) $3\leq x\leq 5$

(B) $3<x\leq 5$

(C) $x<3$ or $x\geq 5$

(D) $x\leq 3$ or $x\geq 5$

3 $\int \cos^2 4x\,dx=$

(A) $\dfrac{1}{2}x+\dfrac{1}{16}\sin 8x+C$

(B) $\dfrac{1}{2}x-\dfrac{1}{16}\sin 8x+C$

(C) $\dfrac{1}{2}x+\dfrac{1}{8}\sin 8x+C$

(D) $\dfrac{1}{2}x-\dfrac{1}{8}\sin 8x+C$

4 If $P(x)=(x+2)(x+k)$ and if the remainder when $P(x)$ is divided by $(x-1)$ is 12, then:

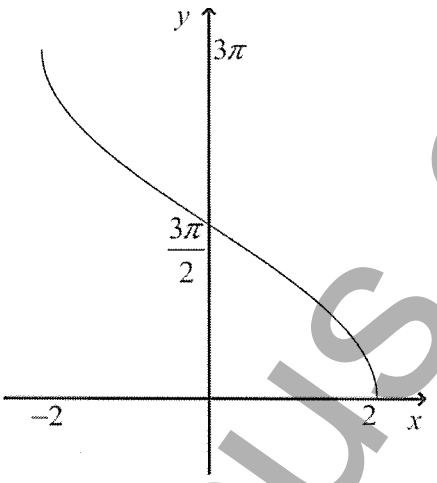
(A) $k=2$

(B) $k=3$

(C) $k=6$

(D) $k=11$

5 Which function best describes the graph below?



(A) $y=2\cos^{-1}3x$

(B) $y=2\cos^{-1}\frac{3x}{2}$

(C) $y=3\cos^{-1}2x$

(D) $y=3\cos^{-1}\frac{x}{2}$

6 If the function f is defined by $f(x)=x^5-1$, then the inverse function of f, is defined by $f^{-1}(x)=$

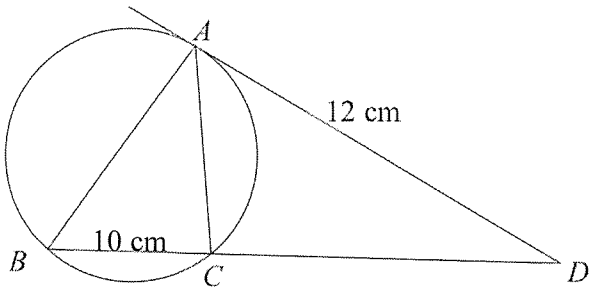
(A) $\sqrt[5]{x}-1$

(B) $\sqrt[5]{x-1}$

(C) $\sqrt[5]{x}+1$

(D) $\sqrt[5]{x+1}$

7 ABC is a triangle inscribed in a circle. The tangent to the circle at A meets BC produced at D where $BC = 10$ and $AD = 12$. What is the length of CD?



NOT TO SCALE

(A) 6 cm

(B) 7 cm

(C) 8 cm

(D) 9 cm

8 $\int \frac{x^2}{e^{x^3}}\,dx =$

(A) $-\frac{1}{3e^{x^3}} + C$

(B) $-\frac{1}{3}e^{x^3} + C$

(C) $-\frac{1}{3}\ln e^{x^3} + C$

(D) $\frac{1}{3}\ln e^{x^3} + C$

9 Consider the curve defined by the parametric equations $x = \frac{1}{t}$ and $y = \frac{t}{t+1}$.

The graph of $y = f(x)$ would have asymptotes:

(A) $x = 0$ only

(B) $x = 1,\ y = -1$

(C) $x = -1$ only

(D) $x = -1,\ y = 0$

10 The velocity, v metres per second, of a particle moving in simple harmonic motion along the x axis is given by the equation $v^2 = 36 - 4x^2$.

What is the amplitude, in metres of the motion of the particle?

(A) 3

(B) 2

(C) 6

(D) 4

End of Section I

Section II

60 marks
Attempt Questions 11 – 14
Allow about 1 hour and 45 minutes for this section

Answer each question in a new writing booklet. Extra writing booklets are available.

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations

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Question 11 (15 marks) Start a new writing booklet

(a) Evaluate $\int_0^2 \frac{dx}{\sqrt{16-x^2}}$. **2**

(b) Differentiate $3x^2 \ln x$, for $x > 0$. **2**

(c) Find the acute angle between the lines $x+2y-5=0$ and $y=4x+5$, giving your answer correct to the nearest minute. **3**

(d) Use the substitution $u = e^x$ to find $\int \frac{e^x}{1+e^{2x}} dx$. **3**

(e) The staff in a school office consists of 5 males and 8 females. **2**

How many committees of 5 staff can be chosen that contain exactly 3 females?

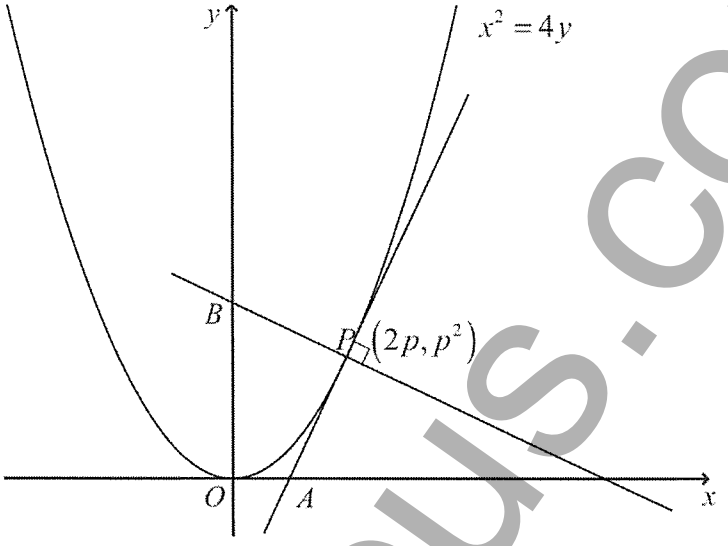
(f) Use the binomial theorem to find the term independent of x in the expansion of **3**

$\left(2x - \frac{1}{x^2}\right)^{12}$.

Question 12 (15 marks) Start a new writing booklet

(a) Use mathematical induction to prove that $n! > 2^n$ for integer $n \geq 4$. **3**

(b) The diagram below shows the graph of the parabola $x^2 = 4y$.
The tangent cuts the parabola at $P(2p, p^2)$, $p > 0$, cuts the x axis at A.
The normal to the parabola at P cuts the y axis at B.



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(i) Show that the equation of the tangent AP is $y = px - p^2$. **2**

(ii) Show that the equation of the normal PB is $x + py = p^3 + 2p$. **1**

(iii) Find the coordinates of A and B. **2**

(iv) Let C divide the interval AB in the ratio 2:1. **3**

Find the Cartesian equation of the locus of C, giving any domain restrictions.

(c) Consider the function $f(x) = 1 + \cos^{-1}(2x-1) - 2\cos^{-1}\sqrt{x}$ for $0 \leq x \leq 1$.

(i) Show that $f'(x) = 0$ for $0 \leq x \leq 1$. **3**

(ii) Sketch the graph of $y = f(x)$ for $0 \leq x \leq 1$. **1**

Question 13 (15 marks) Start a new writing booklet

(a) A particle is moving in simple harmonic motion has its acceleration given by $\frac{d^2x}{dt^2} = -25x$, where x metres is the displacement of the particle after t seconds.

Initially, the particle's acceleration is $50\ \text{ms}^{-2}$ and after $\frac{\pi}{6}$ seconds, the particle's velocity is -10ms^{-1}.

(i) Find the period of the motion. **1**

(ii) Show that $x = a\sin(5t - \alpha)$ is a possible equation of motion for this particle, where a and α are positive constants and α is acute. **2**

(iii) Show that the amplitude of the motion is 4 metres. **2**

(iv) Find the value of α. **1**

(v) Find the greatest speed of the particle and where the particle reaches this speed. **2**

(vi) How many times does the particle change direction in the first 2 seconds? **2**

(b) Let $(2+3x)^7 = \sum_{k=0}^{7} t_k x^k$

(i) Write down an expression for t_k. **1**

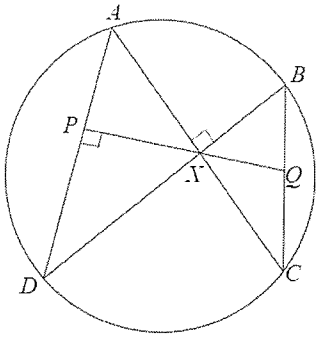
(ii) Hence show that $\frac{t_{k+1}}{t_k} = \frac{21-3k}{2k+2}$ where $0 < k < 7$. **2**

(iii) Hence, or otherwise, find the greatest coefficient in the expansion of $(2+3x)^7$. **2**

Question 14 (15 marks) Start a new writing booklet

(a) The diagram below shows points A, B, C and D on a circle. The lines AC and BD are perpendicular and meet at X.

The perpendicular to AD through X meets AD at P and BC at Q.



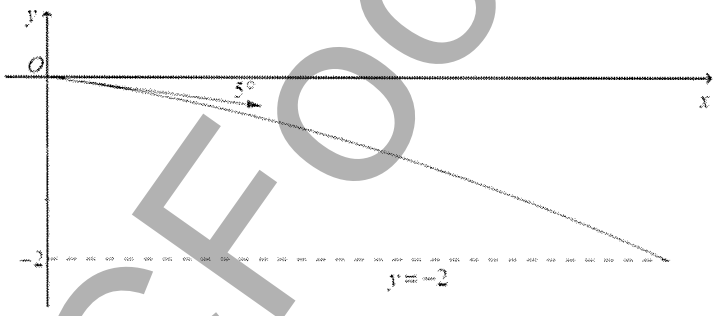
NOT TO SCALE

Copy or trace this diagram into your writing booklet.

(i) Prove that $\angle QXB = \angle QBX$. **3**

(ii) Prove that PQ bisects BC. **2**

(b) A cricket ball leaves a bowler's hand 2 metres above the ground with a velocity of $30\ \text{ms}^{-1}$ at an angle of projection of $5°$ **below** the horizontal, as shown below.



NOT TO SCALE

Using the origin as the point where the ball leaves the bowlers hand, the coordinates of the ball at time t are given by:

$x = 30t\cos 5°$

$y = -30t\sin 5° - 5t^2$

(Do not prove these results)

(i) Find the time it takes for the ball to strike the ground. **2**

(ii) Calculate the angle at which the ball strikes the ground. **2**

(iii) Show the motion of the ball is parabolic, even though it is projected at an angle below the horizontal. **2**

Question 14 continues on page 11

Question 14 (continued)

(c) A television satellite tower stands on a large area of flat ground.

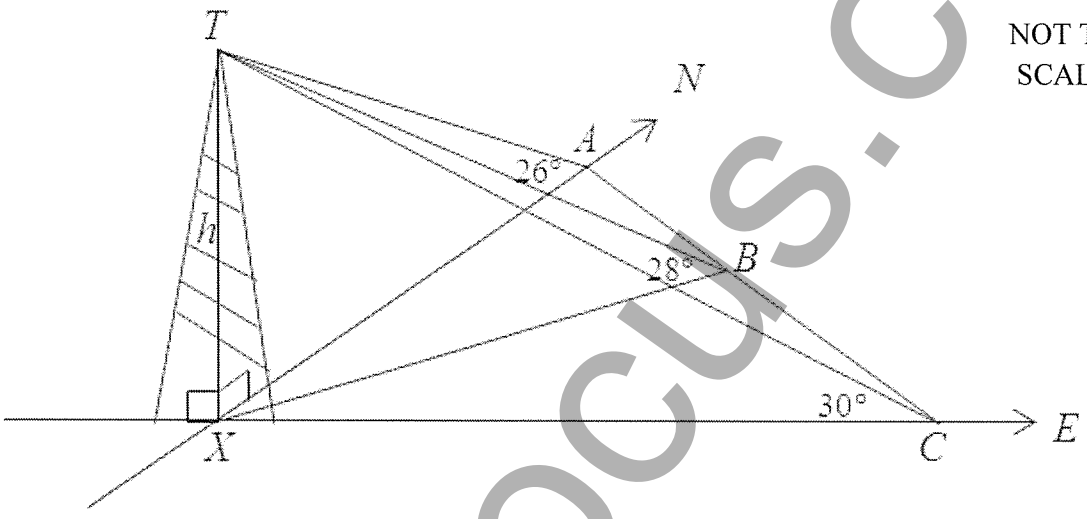
Three maintenance workers, A, B and C, are observing the tower.

Worker A is due north of the tower.

Worker C is due East of the tower.

Worker B is on the line of sight-from A to C (A, B and C are collinear).

The angles of elevation of the top of the tower from A, B and C are 26°, 28° and 30° respectively.



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(i) Find $\angle XAC$, correct to one decimal place. **1**

(ii) Find $\angle ABX$, correct to the nearest degree. **2**

(iii) Hence, find the bearing of Worker B from the base of the tower X, correct to the nearest degree. **1**

End of Paper

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Ext 1 Trial Solutions 2013 HGHS

Multiple Choice

1. $-\frac{b}{a} = -3 \quad \frac{c}{a} = -2 \quad -\frac{d}{a} = 4$

$\therefore a = 1 \quad b = 3 \quad c = -2 \quad d = -4$ (B)

2. $\frac{4}{x-3} \leq 2$

$4(x-3) \leq 2(x-3)^2$

$4(x-3) - 2(x-3)^2 \leq 0$

$(x-3)[4 - 2(x-3)] \leq 0$

$(x-3)(10-2x) \leq 0$

$2(x-3)(5-x)$

$x < 3, \; x \geq 5$ $\therefore$ (C)

3. $\int \cos^2 4x \, dx = \frac{1}{2}\int [\cos 8x + 1] \, dx$

$= \frac{1}{2}\left[\frac{\sin 8x}{8} + x\right]$

$= \frac{x}{2} + \frac{\sin 8x}{16} + c$

$\therefore$ (A)

4. $P(1) = 12$

$\therefore (1+2)(1+k) = 12$

$3 + 3k = 12$

$3k = 9$ $\therefore$ (B)

$k = 3$

5. $-1 \leq \frac{x}{2} \leq 1$

$-2 \leq x \leq 2$ $\therefore$ (D) is only option

6. $f(x) = x^5 - 1$

$\therefore$ for inverse

$x = f(x)^5 - 1$

$\therefore f(x)^5 = x + 1$

$\therefore f(x) = \sqrt[5]{x+1}$ $\therefore$ (D)

7. $AD^2 = BD \times CD$

$12^2 = (10+x)x$

$144 = 10x + x^2$

$0 = x^2 + 10x - 144$

$0 = (x+18)(x-8)$ $\therefore$ (C)

$\therefore x = -18, 8$

$x = 8$ as $x > 0$

8. $\int \frac{x^2}{e^{x^3}} dx = \int x^2 e^{-x^3} dx$

$= -\frac{1}{3} e^{-x^3} + c$

$= -\frac{1}{3e^{x^3}} + c$ $\therefore$ (A)

9. $x = \frac{1}{t}$ then $y = \frac{1}{x}$

$= \frac{1}{\frac{1}{x}+1}$

$= \frac{1}{1+x}$

$\therefore x \neq -1$ and $y = 0$ $\therefore$ (D)

10. $v^2 = 36 - 4x^2$

$= 4(9 - x^2)$

$= n^2(a^2 - x^2)$ $\therefore$ (A)

where $n = 2$ and $a = 3$

Question 11

(a) $\int_0^2 \frac{dx}{\sqrt{16-x^2}} = \left[\sin^{-1}\frac{x}{4}\right]_0^2$

$= \sin^{-1}\frac{1}{2} - \sin 0$

$= \frac{\pi}{6} - 0$

$= \frac{\pi}{6}$

(b) $\frac{d}{dx}\left(3x^2 \ln x\right) = 3x^2 \times \frac{1}{x} + 6x \times \ln x$

$= 3x + 6x \ln x$

OR

$= 3x(1 + 2\ln x)$

(c) $x + 2y - 5 = 0$ and $4x - y + 5 = 0$

$m_1 = -\frac{1}{2}$ $\quad m_2 = 4$

$\therefore \tan\alpha = \left|\frac{4 - -\frac{1}{2}}{1 + 4 \cdot -\frac{1}{2}}\right|$

$= \left|\frac{\frac{9}{2}}{-1}\right|$

$= 4\frac{1}{2}$

$\alpha = 77°28'$

(d) $\int \frac{e^x}{1+e^{2x}} dx = \int \frac{u}{1+u^2} \cdot \frac{du}{u}$

$= \tan^{-1} u + c$

$= \tan^{-1} e^x + c$

$u = e^x$

$\therefore \frac{du}{dx} = e^x$

$dx = \frac{du}{e^x}$

$= \frac{du}{u}$

(e) ${}^{8}C_{3} \times {}^{5}C_{2} = \frac{8\times7\times6}{3\times2\times1} \times \frac{5\times4}{2\times1}$

$= 560$

$\therefore$ 560 different committees of 5 containing exactly 3 females could be chosen.

(f) $\left(2x - \frac{1}{x^2}\right)^{12}$ has general term ${}^{12}C_{r}(2x)^{12-r}\left(-\frac{1}{x^2}\right)^{r}$

$\therefore$ power of x function $= 12-r-2r$

$= 12-3r$

$\therefore$ constant term has power of zero

$\therefore 12-3r = 0$

$\therefore r = 4$

If $r=4$ ${}^{12}C_{4}(2x)^{12-4}\left(-\frac{1}{x^2}\right)^{4} = 495\times2^{8}\times x^{8}\times\frac{1}{x^8}$

$= -126\,720$

Question 12

(a) RTP $n! > 2^n$ for $n \geq 4$

Step 1 Prove for $n=4$

LHS $= 4!$ $= 24$ RHS $= 2^4$ $= 16$ $\therefore$ true for $n=4$

Step 2 If result is true for $n=k$

then $k! > 2^k$

Step 3 RTP for $n=k+1$ ie. $(k+1)! > 2^{k+1}$

LHS $= (k+1)!$

$= (k+1)k!$

$> (k+1)2^k$

$= k.2^k + 2^k$ for $k>4$

$> 2^k + 2^k$

$= 2.2^k$

$\therefore (k+1)! > 2^{k+1}$ for $k \geq 4$

Step 4 Since the result is true for $n=4$, it is therefore true for $n=5$ Hence since it is true for $n=5$ it is true for $n=6$ and so on.

Therefore by the process of Mathematical Induction it is true for all integer k values greater than 4

(b) $x^2 = 4y$ $x = 2p$, $y = p^2$

(i) $\frac{dx}{dp} = 2$ $\frac{dy}{dp} = 2p$ $\therefore \frac{dy}{dx} = \frac{2p}{2} = p$

$\therefore$ Tangent $y - p^2 = p(x-2p)$

$y - p^2 = px - 2p^2$

$y = px - p^2$ as required

(ii) Normal $y - p^2 = -\frac{1}{p}(x-2p)$

$py - p^3 = -x + 2p$

$x + py = p^3 + 2p$ as required

(iii) A $(y=0)$ on tangent $0 = px - p^2$

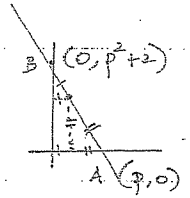
$px = p^2$

$x = p$ $\therefore A(p,0)$

B $(x=0)$ on normal $py = p^3 + 2p$

$y = p^2 + 2$ $\therefore B(0, p^2+2)$

(iv) $A(p,0)$, $B(0,p^2+2)$



$\therefore C\left(\frac{p}{3}, \frac{2p^2+4}{3}\right)$ By the geometry

$\therefore x = \frac{p}{3}$ into $y = \frac{2(3x)^2+4}{3}$

$= \frac{18x^2+4}{3}$

$y = 6x^2 + \frac{4}{3}$

$\therefore C$ is a parabola with Domain $x>0$ (Since $p>0$) and range $y \geq 4/3$, Vertex $(0, 4/3)$

(c)(i) $f(x) = \cos^{-1}(2x-1) - 2\cos^{-1}\sqrt{x} + 1 \qquad 0 \leq x \leq 1$

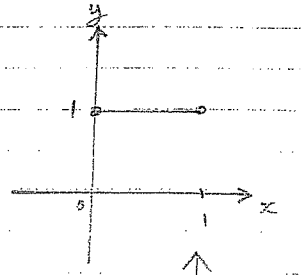
$f'(x) = \frac{-1}{\sqrt{1-(2x-1)^2}} \times 2 - \frac{2}{\sqrt{1-(\sqrt{x})^2}} \times \frac{1}{2\sqrt{x}}$

$= \frac{-2}{\sqrt{1-(4x^2-4x+1)}} + \frac{1}{\sqrt{1-x^2}} \times \frac{1}{\sqrt{x}}$

$= \frac{-2}{\sqrt{4x-4x^2}} + \frac{1}{\sqrt{x}\sqrt{1-x^2}}$

$= \frac{-2}{2\sqrt{x}\sqrt{1-x^2}} + \frac{1}{\sqrt{x}\sqrt{1-x^2}}$

$= 0$ as required



(ii) $f(x)$ is horizontal since $f'(x)=0$ for $0 \leq x \leq 1$

$f(0) = \cos^{-1}(-1) - 2\cos^{-1}(0) + 1$

$= \pi - 2 \times \frac{\pi}{2} + 1 = 1$ (5)

Question 13

(a) $\ddot{x} = -25x$

(i) period $= \frac{2\pi}{n} = \frac{2\pi}{5}$ seconds

(ii) $x = a\sin(5t-\alpha)$

$\dot{x} = 5a\cos(5t-\alpha)$

$\ddot{x} = -25a\sin(5t-\alpha)$

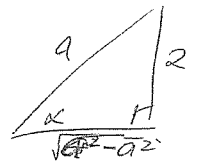
$= -5^2(\sin(5t-\alpha)$

$= -5^2 x$.

ii) $\dot{x} = -25a\sin(-\alpha) = 50$

$\sin(-\alpha) = \frac{-2}{a}$

$\sin\alpha = \frac{2}{a}$ ... (1)



$\dot{x} = 5a\cos\left(\frac{5\pi}{6} - \alpha\right) = -10$

$a\cos\left(\frac{5\pi}{6} - \alpha\right) = -2$ (2)

$a\left[\cos\frac{5\pi}{6}\cos\alpha + \sin\frac{5\pi}{6}\sin\alpha\right] = -2$

$a\left[-\frac{\sqrt{3}}{2} \times \frac{\sqrt{a^2-4}}{a} + \frac{1}{2} \times \frac{2}{a}\right] = -2$

$-\frac{\sqrt{3}}{2a}\sqrt{a^2-4} + \frac{1}{a} = \frac{-2}{a}$

$-\frac{\sqrt{3}}{2}\sqrt{a^2-4} + 1 = -2$

$\sqrt{a^2-4} = \frac{-6}{-\sqrt{3}}$

$a^2 - 4 = \frac{36}{3}$

$a^2 - 4 = 12$

$a^2 = 16$

$a = 4 \quad (a>0)$

) From ①

$\sin\alpha = \frac{2}{a}$

$\sin\alpha = \frac{2}{4}$

$\sin\alpha = \frac{1}{2}$

$\alpha = \pi/6$

) max speed at centre of motion:

$\dot{x} = 20\cos(5t - \pi/6)$

$-1 \leq \cos(5t - \pi/6) \leq 1$

$20 \leq 20\cos(5t - \pi/6) \leq 20$

∴ max speed is 20ms^{-1} and happens at $x = 0$.

) changes direction when $\dot{x} = 0$

$5t - \pi/6 = \pi/2, 3\pi/2, 5\pi/2, 7\pi/2, \ldots$

$5t = 4\pi/6, 10\pi/6, 16\pi/6, 22\pi/6$

$t = \frac{2\pi}{15}, \frac{5\pi}{15}, \frac{8\pi}{15}, \frac{11\pi}{15}$

$\doteq \underline{0.4}, \underline{1.04}, \underline{1.67}, 2.3$

∴ Changes direction three times in first two seconds

(b) (i) $(2+3x)^7 = \sum_{k=0}^{7} {}^7C_k (2)^{7-k}(3x)^k$.

$\therefore t_k = {}^7C_k\, 2^{7-k}.3^k$.

(ii) $t_{k+1} = {}^7C_{k+1}\, 2^{6-k}.3^{k+1}$.

$\frac{t_{k+1}}{t_k} = \frac{7! \times 2^{6-k}.3^{k+1}}{(6-k)!(k+1)!} \times \frac{(7-k)!\,k!}{7!\,2^{7-k}.3^k}$.

$= \frac{3 \times (7-k)}{2 \times (k+1)}$

$= \frac{21-3k}{2k+2}$.

(iii) $\frac{21-3k}{2k+2} \geq 1$

$21 - 3k \geq 2k + 2 \quad (2k+2 \geq 0)$

$5k \leq 19$

$k \leq 3\frac{4}{5}$

∴ $k = 3$ gives largest coefficient

$t_{3+1} = t_4 = {}^7C_4 \cdot 2^3 \cdot 3^4$

$= 22680$.

QUESTION 14

i) Let $\angle QBX = x^\circ$

$\angle QAD = x^\circ$ ($\angle$'s in same segment)

$\angle AXA + 90^\circ + x^\circ = 180^\circ$ ($\angle$ sum $\triangle AXA$)

$\angle AXP = 90^\circ - x^\circ$

$\angle AXA + \angle AXB + \angle BXQ = 180$ (straight line PBQ)

$90 - x + 90 + \angle BXQ = 180$

$\angle BXQ = x$

$\therefore \angle BXQ = \angle QAX$ (both equal x)

ii) $\angle DAX + \angle AXD + \angle XDA = 180$ ($\angle$ sum $\triangle AXD$)

$x + 90 + \angle XDA = 180$

$\angle XDA = 90 - x$

$\angle XDA = \angle XCB = 90 - x$ ($\angle$'s in same segment)

$\angle AXP = \angle CXQ = 90 - x$ (vertically opposite $\angle$'s)

$\therefore \angle CXQ = \angle XCB$ (both equal $90 - x$)

$\therefore \triangle QCX$ is isosceles

& $\triangle BQX$ is isosceles

$\therefore BQ = XQ$ ($\angle$'s opposite equal sides)

$XQ = CQ$ (" " " ")

$\therefore BQ = CQ$ (both equal XQ)

$\therefore PQ$ bisects BC.

b) i) when $y = -2$

$-2 = -30t\sin 5 - 5t^2$

$5t^2 + 30t\sin 5 - 2 = 0$

$t = \dfrac{-30\sin 5 \pm \sqrt{900\sin^2 5 + 40}}{10}$

$= 0.423$

$= 0.4$.

ii) $\dot{x} = 30\cos 5$

$\dot{y} = -30\sin 5 - 10t$

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx}$

$\tan\theta = \dfrac{-30\sin 5 - 10t}{30\cos 5}$

$= -0.2213$

$\theta = -12.48^\circ$

$\therefore \theta = 12.48^\circ$ or 167.52°

iii) $x = 30t\cos 5$ — ①

$y = -30t\sin 5 - 5t^2$ — ②

from ① $\Rightarrow t = \dfrac{x}{30\cos 5}$ — ③

subst ③ into ②

$y = -\dfrac{30x\sin 5}{30\cos 5} - \dfrac{5x^2}{900\cos^2 5}$

$= \dfrac{-x^2}{180\cos^2 5} - x\tan 5$

of the form $y = ax^2 + bx + c$

$\therefore$ parabolic

c) $\tan 26 = \dfrac{h}{XA}$

$XA = \dfrac{h}{\tan 26}$

$\tan 30 = \dfrac{h}{XC}$

$XC = \dfrac{h}{\tan 30}$

$\tan 28 = \dfrac{h}{XB}$

$XB = \dfrac{h}{\tan 28}$

i) $\tan \angle XAC = \dfrac{XC}{XA}$

$= \dfrac{h}{\tan 30} \times \dfrac{\tan 26}{h}$

$= 0.84478$

$\angle XAC = 40.2^\circ$

ii) $\dfrac{\sin \angle ABX}{XA} = \dfrac{\sin 40.2}{XB}$

$\sin \angle ABX = \dfrac{\sin 40.2 \times \tan 28}{\tan 26}$

$= 0.7037$

$\therefore \angle ABX = 45^\circ, 135^\circ$

$\therefore \angle ABX = 135^\circ$ as $\angle ABX > 49.8^\circ$

iii) $180 - 135 - 40 = 005^\circ T$ or $N5^\circ E$