Question 11.

a. $\dfrac{2x+1}{1-x} \geq 1$

various methods

example

$\dfrac{2x+1-1+x}{1-x} \geq 0$ ✓working

$(1-x)^2 \times \dfrac{3x}{\cancel{1-x}} \geq 0 \times (1-x)^2$

$(1-x)(3x) \geq 0$

$0 \leq x \leq 1$

but $x \neq 1$ ✓

$\therefore 0 \leq x < 1$ ✓

③

b. i) amplitude $= 2$ ✓

ii) period $= \dfrac{2\pi}{3}$ ✓

②

c. $CP \times DP = AP \times BP$

$CP \times 5 = 10 \times 3$

$CP = \underline{6\text{cm}}$ ✓

①

d. Domain: $-1 \leq 2x - 1 \leq 1$

$0 \leq x \leq 1$

Range: $-\dfrac{\pi}{2} \leq \dfrac{y-1}{2} \leq \dfrac{\pi}{2}$

$1-\pi \leq y \leq 1+\pi$

$\therefore a = 1$ ✓

$b = 1-\pi \qquad c = 1+\pi$ ✓ for both

②

e. $\dfrac{x}{3} = \tan\theta \qquad \dfrac{y}{2} = \sec\theta$ ✓ working

connector: $\tan^2\theta + 1 = \sec^2\theta$

$\therefore \dfrac{x^2}{9} + 1 = \dfrac{y^2}{4}$ ✓

or $4x^2 - 9y^2 = -36$

②

f. $\sin 2\theta + \cos\theta = 0 \qquad 0 \leq \theta \leq 2\pi$ ✓

$2\sin\theta\cos\theta + \cos\theta = 0$

$\cos\theta(2\sin\theta + 1) = 0$

$\therefore \cos\theta = 0$ ✓ $\qquad \sin\theta = -\dfrac{1}{2}$ ✓

$\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2} \qquad \theta = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

$\therefore \theta = \dfrac{\pi}{2}, \dfrac{7\pi}{6}, \dfrac{3\pi}{2}, \dfrac{11\pi}{6}$

③

g. $V = \frac{4}{3}\pi r^3 \qquad \dfrac{dr}{dt} = 2\text{cm/sec}$

$\dfrac{dV}{dt} = \dfrac{dV}{dr} \times \dfrac{dr}{dt}$

$= 4\pi r^2 \times 2$ ✓

at $r = 10$

$\dfrac{dV}{dt} = 800\pi \text{ cm}^3/\text{sec}$ ✓

②

/15

Multiple choice

1. A	6. C
2. C	7. A
3. C	8. B
4. A	9. C
5. C	10. D

Question 12

a. $\displaystyle\int_0^{\frac{\pi}{6}} 2\sin^2 x\,dx = \int_0^{\frac{\pi}{6}} (1-\cos 2x)\,dx$ ✓

$= \left[x - \tfrac{1}{2}\sin 2x\right]_0^{\frac{\pi}{6}}$

$= \left(\dfrac{\pi}{6} - \dfrac{1}{2}\sin\dfrac{\pi}{3} - 0\right)$ ✓

$= \dfrac{\pi}{6} - \dfrac{\sqrt{3}}{4}$ or $\dfrac{2\pi - 3\sqrt{3}}{12}$

②

b. $\displaystyle\int_1^4 \dfrac{dx}{x+\sqrt{x}} \qquad u = x^{\frac{1}{2}} \qquad$ at $x=4 \quad u=2$

$\dfrac{du}{dx} = \dfrac{1}{2}x^{-\frac{1}{2}}$ ✓ $\qquad x=1 \quad u=1$

$2u\,du = dx$.

$= \displaystyle\int_1^2 \dfrac{2u\,du}{u^2+u}$

$= \displaystyle\int_1^2 \dfrac{2\,du}{u+1}$ ✓

$= \left[2\ln(u+1)\right]_1^2$

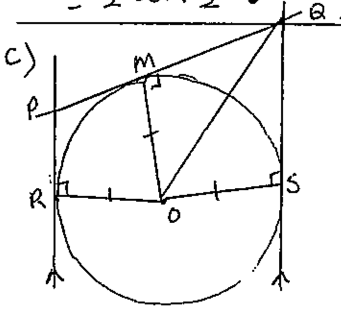
$= 2\ln 3 - 2\ln 2$

$= 2\ln\dfrac{3}{2}$ ✓

③

Suggested solutions and marking scale for Trial yr12 Ext1 2013.

• variations do occur

c) 

ii) Aim: prove $\triangle MOQ \equiv \triangle SOQ$.

Method:

In $\triangle MOQ$ and $\triangle SOQ$.

$MO = OS$ (radii)

OQ is common

$Q\hat{M}O = Q\hat{S}O$ (radii to tangent is ⊥)

$= 90°$

$\therefore \triangle MOQ \equiv \triangle SOQ$ (R.H.S)

② marks – complete proof

① mark – 2 reasons.

②

Quest 12 Cont.

iii) An example

Similar to (ii) $\triangle MOP \equiv \triangle ROP$ (R.H.S)

let $M\hat{O}P = y$

$= R\hat{O}P$ (matching $\angle$s in congruent $\triangle$s)

and let $R\hat{P}O = z$

$= M\hat{P}O$ (matching $\angle$s in congruent $\triangle$s)

Since tangent PQ meets RP and SQ (parallel tangents)

$R\hat{P}Q + S\hat{Q}M = 180$ (cointerior $\angle$s, $PR \parallel QS$)

$\therefore 2x + 2y = 180$

$x + y = 90$ --- (1)

now $P\hat{O}M = 90 - x$ ($\angle$ sum of $\triangle$ POM)

$Q\hat{O}M = 90 - y$ ($\angle$ sum of $\triangle$ QOM)

then $P\hat{O}Q = 90 - x + 90 - y$ (addition of adjacent $\angle$s)

$= 180 - (x+y)$

using (1) $= 180 - 90$

$= 90^\circ$ as required.

* needed to establish ROS was a straight line!

* various methods. (1) mark connecting (ii)

(2) marks - clear proof

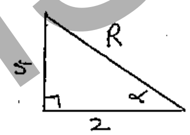
(2)

d) $x = 2\cos 3t - 5\sin 3t$

$x = R\cos 3t \cos\alpha - R\sin 3t \sin\alpha$.

$R\cos\alpha = 2$ $\quad$ $R\sin\alpha = 5$

$\cos\alpha = \frac{2}{R}$ $\quad$ $\sin\alpha = \frac{5}{R}$.



$R = \sqrt{29}$ ✓

$\tan\alpha = \frac{5}{2}$

$\alpha = 1.19$ (2dp)

or $= \tan^{-1}\left(\frac{5}{2}\right)$

$\therefore x = \sqrt{29}\cos(3t + 1.19)$ ✓ or $\sqrt{29}\cos\left(3t + \tan^{-1}\frac{5}{2}\right)$

(2)

Question 12 Cont.

d (ii) $x = \sqrt{29}\cos(3t + 1.19)$

$\dot{x} = -3\sqrt{29}\sin(3t + 1.19)$

$\ddot{x} = -9\sqrt{29}\cos(3t + 1.19)$

✓ $= -9 \times x$ $\quad$ now defn of SHM is $\ddot{x} = -n^2 x$

$\therefore$ the particle is moving in SHM with $n = 3$.

(1)

e) $\left(ax - \frac{b}{x^2}\right)^5 = (ax - bx^{-2})^5$

$T_{k+1} = {}^nC_k\, a^{n-k} \cdot b^k$ for $(a+b)^n$

$= {}^5C_k (ax)^{5-k} \cdot (-1)^k \cdot (bx^{-2})^k$ ✓

$= {}^5C_k\, a^{5-k} \cdot (-1)^k \cdot b^k \cdot x^{5-k} \cdot x^{-2k}$

to find k.

for x^2: $2 = 5 - 3k$
$3k = 3$
$k = 1$

Coef of $T_2 = {}^5C_1 a^4 (-b)^1$
$= -5a^4 b$

for x^{-1}: $-1 = 5 - 3k$
$3k = 6$
$k = 2$

Coef of $T_3 = {}^5C_2 a^3 b^2$
$= 10a^3 b^2$ ✓

since coefs of x^2 and x^{-1} are equal

$\therefore -5a^4 b = 10a^3 b^2$

$10a^3 b^2 + 5a^4 b = 0$ ✓

$5a^3 b(2b + a) = 0$

$\therefore 2b + a = 0$ as required.

(3)

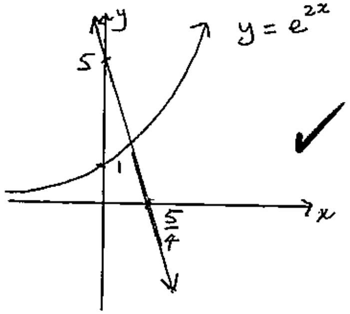
Question 13.

a) no. of ways $= {}^9C_5 - {}^7C_3$
$= 126 - 35$
$= 91$ ✓

(2)

b).

i)



$e^{2x} + 4x - 5 = 0$
let $y = e^{2x} = 5 - 4x$.
$\therefore y = e^{2x}$
$y = 5 - 4x$

✓

or 1 mark for the correct graph of $y = e^{2x} + 4x - 5$

ii) $f(x) = e^{2x} + 4x - 5$

$f(0) = e^0 - 5$
$= -4 < 0$ ✓

$f(1) = e^2 + 4 - 5$
$= 6.39 > 0$

$\therefore$ a root lies between 0 and 1 as $f(x)$ neg to post

iii) let $x_1 = 0.5$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 0.5 - \frac{-0.2817\ldots}{9.436\ldots}$ ✓

$= 0.529\ldots$
$= 0.53$ (2dp) ✓

$f(0.5) = -0.2817\ldots$
$f'(x) = 2e^{2x} + 4$
$f'(0.5) = 9.436\ldots$

(4)

c. i) $\ddot{x} = 2e^x$ $\quad t = 0, v = -2$ m/sec $\quad x = 0$.

since $\ddot{x} = \frac{d}{dx}(\frac{1}{2}v^2)$

then $\frac{d}{dx}(\frac{1}{2}v^2) = 2e^x$
$\frac{1}{2}v^2 = 2\int e^x dx$
$\frac{1}{2}v^2 = 2e^x + C$

at $v = -2$ $x = 0$
$2 = 2 \times 1 + C$

1 mark working towards

1 mark

$C = 0$ $\quad \therefore v^2 = 4e^x$.

(2)

Quest 13 Cont.

ii) $v^2 = 4e^x$
$v = \pm\sqrt{4e^x}$
$= \pm 2e^{\frac{x}{2}}$

now $v < 0$ initially ✓
$\therefore v = -2e^{\frac{x}{2}}$

In terms of t

$\frac{dx}{dt} = -2e^{\frac{x}{2}}$

$\frac{dt}{dx} = -\frac{1}{2}e^{-\frac{x}{2}}$

$t = e^{-\frac{x}{2}} + C$

at $t = 0$ $x = 0$
$\therefore 0 = e^0 + C$
$C = -1$ ✓
$\therefore t = e^{-\frac{x}{2}} - 1$
$t + 1 = e^{-\frac{x}{2}}$
$\ln(t+1) = -\frac{x}{2}$ (as $\ln e = 1$)

$\therefore x = -2\ln(t+1)$ ✓

(3)

d) an example
i) $V = 500 - 500e^{-kt}$

$\frac{dv}{dt} = k\,500e^{-kt}$

but $\therefore 500e^{-kt} = 500 - V$.

$= k(500 - V)$
$= -k(v - 500)$ as required)

✓ any solid method

ii) $v = 0$ $\quad t = 0$ $\quad V = 21$ m/sec $\quad t = 5$

$V = 500 - 500e^{-kt}$
$21 = 500 - 500e^{-5k}$
$-479 = -500e^{-5k}$
$0.958 = e^{-5k}$
$\ln(0.958) = \ln e^{-5k}$
$\therefore k = \frac{\ln 0.958}{-5}$ ✓
$= 0.008581\ldots$
$= 0.00858$ (3 sig fig)

iii) $v = ?$ when $t = 20$ sec
$V = 500 - 500e^{-20k}$
$= 78.8546$ ✓
$\doteqdot 78.9$ m/sec (1dp)

iv) as $t \to \infty$ $\quad e^{-kt} = \frac{1}{e^{kt}} \to 0$

$\therefore V \to 500$ m/sec. ✓

(4)

Question 14.

a) $x = 30\sqrt{2}\,t \qquad y = 30\sqrt{2}\,t - 5t^2$

i) $x = 120\text{ m} \quad t = ?$

$120 = 30\sqrt{2}\,t$

$t = \frac{4}{\sqrt{2}}$ sec ✓

height: $y = 30\sqrt{2}\,t - 5t^2$

$= 30\sqrt{2}\left(\frac{4}{\sqrt{2}}\right) - 5\left(\frac{4}{\sqrt{2}}\right)^2$

$= 120 - 40$

$= 80\text{ m}$ ✓

ii) horizontal range: $y = 0 \quad t = ?$

$30\sqrt{2}\,t - 5t^2 = 0$

$5t\,(6\sqrt{2} - t) = 0$

$t = 0 \qquad t = 6\sqrt{2}$.

range: $x = 30\sqrt{2} \times 6\sqrt{2}$

$= 360\text{ m}$ ✓

Therefore this particle will travel 360m horizontally.

3

b) i) $P(x_1, y_1) \qquad Q(x_1 + a, y_2)$ since $x^2 = 4ay$

$y = \frac{x^2}{4a}$

$y_2 = \frac{(x_1+a)^2}{4a}$ ✓

midpt C $\left(\frac{x_1 + x_1 + a}{2},\ \frac{\frac{x_1^2}{4a} + \frac{(x_1+a)^2}{4a}}{2}\right)$

$= \left(\frac{2x_1 + a}{2},\ \frac{x_1^2 + x_1^2 + 2ax_1 + a^2}{8a}\right)$ ✓

$= \left(\frac{2x_1 + a}{2},\ \frac{2x_1^2 + 2ax_1 + a^2}{8a}\right)$

2

Quest. 14. Cont.

b ii)

$x = \frac{2x_1 + a}{2} \qquad y = \frac{2x_1^2 + 2ax_1 + a^2}{8a}$

$2x = 2x_1 + a$

$x_1 = \frac{2x - a}{2}$

$y = \frac{2\left(\frac{2x-a}{2}\right)^2 + \cancel{2}a\left(\frac{2x-a}{\cancel{2}}\right) + a^2}{8a}$ ✓

$= \frac{\frac{\cancel{2}(4x^2 - 4ax + a^2)}{\cancel{4}\,2} + 2ax - \cancel{a^2} + \cancel{a^2}}{8a}$

$= \frac{4x^2 - \cancel{4ax} + a^2 + \cancel{4ax}}{16a}$

$= \frac{4x^2 + a^2}{16a}$. ✓

$\therefore 16ay = 4x^2 + a^2$

$4x^2 = 16ay - a^2$

$\therefore$ a parabola (the vertex is $\left(0, \frac{a}{16}\right)$)

2

c.) i) $\sum_{r=1}^{n} r^3 = \frac{1}{4} n^2 (n+1)^2$

test true for $n=1$: LHS $= 1^3 = 1$ RHS $= \frac{1}{4} \times 1\,(1+1)^2 = \frac{1}{4} \times 4 = 1$

$\therefore$ LHS = RHS — true for $n = 1$

assume true for $n = k$: $\sum_{r=1}^{n} r^3 = 1 + 2^2 + 3^3 + \cdots n^3 = \frac{1}{4}(n^2)(n+1)^2$

$S_k = 1 + 2^2 + 3^3 + \cdots k^3 = \frac{1}{4} k^2 (k+1)^2$

prove true for $n = k+1$: ie. $S_k + T_{k+1} = S_{k+1}$

$S_{k+1} = \frac{1}{4}(k+1)^2(k+2)^2$

$$LHS. = \frac{1}{4}k^2(k+1)^2 + (k+1)^3$$

$$= \frac{(k+1)^2}{4}\left(k^2 + 4(k+1)\right)$$

must have $= \frac{1}{4}(k+1)^2(k+2)^2$

$= RHS.$

* ∴ If true for $n=k$ now proved true for $n=k+1$

Since true for $n=1$ now true for $n=1+1=2$, $n=3$ and so on by the principles of M.I for all n.

3 marks - correct

2 marks - one error 1 Mark - 2 errors.

ii) $y = x^3 + x$

intro $h = \left(\frac{1}{n}\right)^3 + \frac{1}{n}$.

width $= \frac{1}{n}$

$$Area = \frac{1}{n} \times \left(\frac{1}{n^3} + \frac{1}{n}\right) ✓$$

2nd strip $h = \left(\frac{2}{n}\right)^3 + \frac{2}{n}$

$A = \frac{1}{n}\left(\left(\frac{2}{n}\right)^3 + \frac{2}{n}\right)$

last strip $h = \left(\frac{n-1}{n}\right)^3 + \frac{n-1}{n}$

$A = \frac{1}{n} \times \left(\left(\frac{n-1}{n}\right)^3 + \frac{n-1}{n}\right)$

✓working towards An.

$$∴ \text{Total Area } A_n = \frac{1}{n} \times \left[\left(\frac{1}{n^3} + \frac{1}{n}\right) + \left(\left(\frac{2}{n}\right)^3 + \frac{2}{n}\right) + \left(\left(\frac{3}{n}\right)^3 + \frac{3}{n}\right) + \dots + \left(\left(\frac{n-1}{n}\right)^3 + \frac{n-1}{n}\right)\right]$$

$$∴ \text{two parts } A_n = \frac{1}{n}\sum_{r=1}^{n-1}\left(\frac{r}{n}\right)^3 + \frac{1}{n}\sum_{r=1}^{n-1}\frac{r}{n} \quad ✓ \text{splitting}$$

$$= \frac{1}{n^4}\sum_{r=1}^{n-1}(r)^3 + \frac{1}{n^2}\sum_{r=1}^{n-1} r$$

iii) Area $= \lim_{n\to\infty} A_n$ and $\sum_{r=1}^{n} r^3 = \frac{1}{4}(n)^2(n+1)^2$ from (i)

$$A_n = \frac{n^2(n+1)^2}{4} \times \frac{1}{n^4} + \frac{1}{n^2}\sum_{r=1}^{n-1} r$$

now $\sum_{r=1}^{n-1} r$ is a series (AP)

$1 + 2 + 3 \dots n-1$

$$∴ S_n = \frac{n}{2}(a+l)$$

$$= \frac{n-1}{2}(1+n-1)$$

$$= \frac{(n-1)}{2}(n)$$

$$∴ A_n = \frac{n^2(n+1)^2}{4n^4} + \frac{1}{n^2} \times \frac{(n-1)}{2}n$$

$$= \frac{1}{4} \cdot \frac{(n+1)^2}{n^2} + \frac{1}{2} \cdot \frac{(n^2-n)}{n^2}$$

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty}\left[\frac{1}{4}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) + \frac{1}{2}\left(1 - \frac{1}{n}\right)\right]$$

$$= \frac{1}{4} + \frac{1}{2} \quad ✓✓ \quad \text{as } \frac{1}{n}, \frac{1}{n^2} \to 0 \text{ as } n \to \infty$$

$$= \frac{3}{4} u^2$$

* needed to show working that achieves $\frac{1}{4}$ and $\frac{1}{2}$.