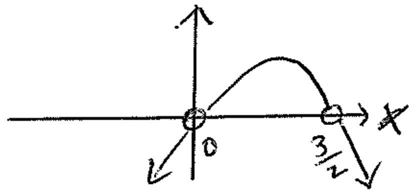
Question 1.

(a) $x(3-2x)>0$



$\therefore \boxed{0 < x < \frac{3}{2}}$

(b)

$$\frac{d}{dx}\left[e^{-x}\cdot\cos^{-1}x\right]$$

$$= e^{-x}\cdot\frac{-1}{\sqrt{1-x^2}} + \cos^{-1}x\cdot -e^{-x}$$

$$= \frac{-e^{-x}}{\sqrt{1-x^2}} - e^{-x}\cos^{-1}x$$

(c) let $P(x)=x^3+ax^2-3x+5$

then $P(-2)=11$ (Rem. Th)

$\Rightarrow (-8)+4a+6+5=11$

$4a=8$

$\boxed{a=2}$

(d) $2\cos x+\sqrt{3}=0$

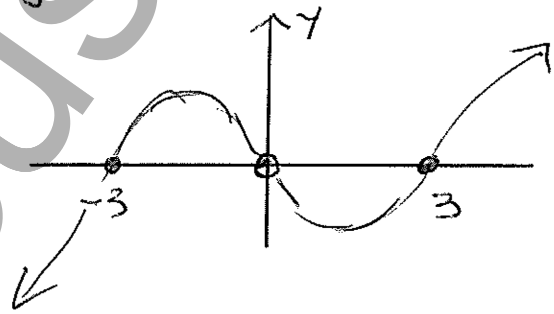
$\Rightarrow \cos x = -\frac{\sqrt{3}}{2}$

ie $\cos x = \cos\frac{5\pi}{6}$

$\therefore \boxed{x = 2n\pi \pm \frac{5\pi}{6}}$

(e) $\frac{x^2-9}{x} \geq 0 \quad [x \neq 0]$

$\times$ by $x^2 \Rightarrow x(x^2-9)\geq 0$



$\boxed{-3 \leq x < 0 \quad \text{or} \quad x \geq 3}$

(f) $\int_0^2 \frac{dx}{4+x^2}$

$$= \frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_0^2$$

$$= \frac{1}{2}\left[\tan^{-1}1 - \tan^{-1}0\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4}-0\right]$$

$$= \frac{\pi}{8}$$

QUESTION 2

(a) $x = \ln u$ $\quad \frac{du}{dx} = e^x$

$u = e^x$

$dx = \frac{du}{e^x}$

$= \frac{du}{u}$

$$\int \frac{\cancel{u}}{\sqrt{1-u^2}} \frac{du}{\cancel{u}}$$

$= \sin^{-1} u + c$

$= \sin^{-1} e^x + c$

(b) $\cos x - x = 0$

$f(x_1) = \cos 0.5 - 0.5 = 0.378$

$f'(x_1) = -\sin 0.5 - 1 = -1.479$

$x_2 = 0.5 - \frac{0.378}{-1.479}$

$= 0.7556$

$= 0.76$ 2 d.p.

(c) $m_1 = 2e^{2x}$ $m_2 = 4 - 2x$

$x = 0$ $m_1 = 2$, $m_2 = 4$

$\tan\theta = \left| \frac{2 - 4}{1 + 2.4} \right|$

$= 2/9$

$\theta = 12°32'$

(d) (i) ${}^6C_2 \times {}^7C_2 \times 3 = 945$

(ii) ${}^5C_1 \times {}^6C_1 \times 3$

$= 90$

3 unit Trial ASC 2009

(3) (a) $\cos\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$

$= \cos\left(-\frac{\pi}{6}\right)$

$= \cos\frac{\pi}{6}$ even fn

$= \frac{\sqrt{3}}{2}.$ (1)

(b) (i) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ (1)

(ii) let $\beta = \alpha$. $\cos(\alpha+\alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

using $\sin^2\alpha + \cos^2\alpha = 1$

$\cos 2\alpha = 1 - \sin^2\alpha - \sin^2\alpha$

$= 1 - 2\sin^2\alpha.$ (1)

(iii) $\lim_{x\to 0} \frac{1-\cos 2x}{x^2} = \lim_{x\to 0} \frac{2\sin^2 x}{x^2} = 2\lim_{x\to 0} \frac{\sin x}{x} \times \frac{\sin x}{x}$

since $\cos 2x = 1 - 2\sin^2 x$

$2\sin^2 x = 1 - \cos 2x$

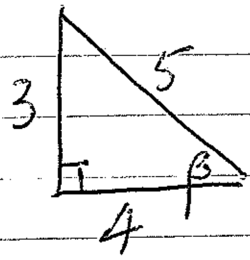
$= 2 \times 1 \times 1$

$= 2.$ (1)

(c) $\alpha = \tan^{-1}\left(\frac{5}{12}\right)$ $\beta = \cos^{-1}\left(\frac{4}{5}\right)$

$\tan\alpha = \frac{5}{12}$ $\cos\beta = \frac{4}{5}$

So $\tan\beta = \frac{3}{4}$



So $\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \frac{\frac{5}{12} - \frac{3}{4}}{1+\frac{5}{12}\times\frac{3}{4}} = \frac{-\frac{1}{3}}{1\frac{5}{16}}$ (2)

$= \frac{-16}{63}$

check $(-1, 7)$ $(5, -2)$ $\underset{m}{17} : \underset{n}{16}$

$$\frac{17\times 5 + 16\times -1}{17+16}, \frac{17\times -2 + 16\times 7}{17+16}$$

$$= \frac{69}{33} = 2\frac{1}{11} \checkmark \qquad \frac{78}{33} = 2\frac{4}{11} \checkmark$$

(e) $1\times 1! + 2\times 2! + 3\times 3! + \cdots + n\times n! = (n+1)! - 1$
for n a positive integer

step 1 let $n=1$, $LHS = 1\times 1! = 1$
$RHS = (1+1)! - 1 = 2! - 1 = 2-1 = 1$

So $n=1$ is true.

step 2 Assuming it is true for $n=k$,
$1\times 1! + 2\times 2! + 3\times 3! + \cdots + k\times k! = (k+1)! - 1$
we must prove that for $n=k+1$,
$1\times 1! + 2\times 2! + 3\times 3! + \cdots + k\times k! + (k+1)\times(k+1)!$
$= (k+2)! - 1$

LHS $(k+1)! - 1 + (k+1)(k+1)!$

$= (k+1)!\,[1+k+1] - 1$

$= (k+1)!\,[k+2] - 1$

$= (k+2)! - 1.$

$=$ RHS.

step 3 Hence the statement is true for $n = k+1$
By the principle of math induction it is true for all $n \geq 1$.

3

4) a) $\dfrac{dy}{dx} = 1 + y$

$\dfrac{dx}{dy} = \dfrac{1}{1+y}$

$x = \ln(1+y) + C$

when $x=0$, $y=2$

$0 = \ln(3) + C$

$C = -\ln 3$

$x = \ln(1+y) - \ln 3$

$x = \ln\left(\dfrac{1+y}{3}\right)$

$\dfrac{1+y}{3} = e^x$

$1+y = 3e^x$

$y = 3e^x - 1$

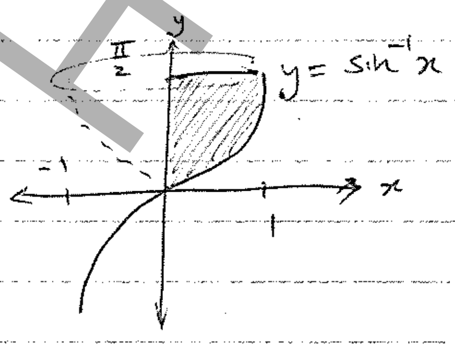
b) let $R\hat{Q}A = x$

$A\hat{B}R = x$ (angles in same segment)

$B\hat{P}A = x$ (alternate segment theorem)

since alternate angles equal ($R\hat{Q}P = Q\hat{P}B$)

$PB \parallel QR$

c) 

$x = \sin y$.

$$V = \pi \int_a^b x^2 \, dy$$

$$V = \pi \int_0^{\frac{\pi}{2}} \sin^2 y \, dy$$

$$V = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} (1 - \cos 2y) \, dy$$

$$V = \frac{\pi}{2} \left[y - \frac{1}{2} \sin 2y \right]_0^{\frac{\pi}{2}}$$

$$V = \frac{\pi}{2} \left[\frac{\pi}{2} - \overset{0}{\cancel{\frac{1}{2} \sin 2\left(\frac{\pi}{2}\right)}} - \left(0 - \overset{0}{\cancel{\frac{1}{2} \sin 2(0)}}\right) \right]$$

$$V = \frac{\pi^2}{4} \text{ units}^3.$$

d) $$a = \frac{1}{(x+3)^2}$$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = (x+3)^{-2}$$

$$\frac{1}{2}v^2 = \frac{(x+3)^{-1}}{-1 \times 1} + C$$

$$\frac{1}{2}v^2 = -\frac{1}{x+3} + C$$

when $x = 0$, $v = 0$

$$0 = -\frac{1}{3} + C$$

$$C = \frac{1}{3}$$

$$\frac{1}{2}v^2 = \frac{1}{3} - \frac{1}{x+3}$$

$$v^2 = 2\left(\frac{1}{3} - \frac{1}{x+3}\right)$$

$$v = \pm\sqrt{2\left(\frac{1}{3} - \frac{1}{x+3}\right)}$$

but acceleration is always positive. & since it starts from rest.

$$v = \sqrt{2\left(\frac{1}{3} - \frac{1}{x+3}\right)} \quad \underline{\text{OR}} \quad \sqrt{\frac{2x}{3(x+3)}}$$

Question 5

(a) (i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right)=\ddot{x}=-3-3x$

$= -3(x+1)$

Let $X=x+1$, so $\ddot{X}=\ddot{x}$

$\therefore \quad \ddot{X}=-3\cancel{x}$ [2]

Hence, Simple Harmonic Motion.

(ii) From above, $n^2=3$

$V^2=3(8-2x-x^2)$

$=3(8-(x^2+2x+1)+1)$

$=3(9-(x+1)^2)$

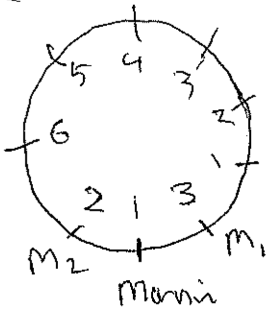
$\therefore v^2=3(9-X^2)$

$\therefore a^2=9 \qquad T=\frac{2\pi}{\sqrt{3}}$ [2]

$a=3$

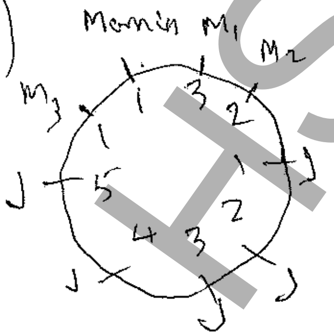
(b) (i) $8!=40320$ ways [1]

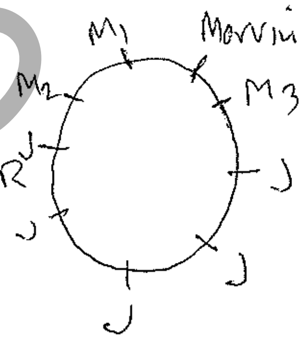
(ii)



$2\times3\times6!=4320$ ways [2]

(iii)





$\therefore 2\times(3\times2\times5!)=1440$ ways

[2]

(c) $x^3+px^2+qx+r=0$

Let roots be $\alpha, \beta, \alpha+\beta$

Now $-p=2(\alpha+\beta)$

$q=\alpha\beta+(\alpha^2+\alpha\beta)+(\alpha\beta+\beta^2)$

$=3\alpha\beta+\alpha^2+\beta^2$

$-r=\alpha\beta(\alpha+\beta)$

$=\alpha^2\beta+\alpha\beta^2$ [1]

RTP: $p^3+8r=4pq$

LHS $=-8(\alpha+\beta)^3+(-8(\alpha^2\beta+\alpha\beta^2))$

$=-(8(\alpha^3+3\alpha^2\beta+3\alpha\beta^2+\beta^3)+8(\alpha^2\beta+\alpha\beta^2))$

$=-(8\alpha^3+32\alpha^2\beta+32\alpha\beta^2+8\beta^3)$

RHS $=-8(\alpha+\beta)(3\alpha\beta+\alpha^2+\beta^2)$

$=-8(3\alpha^2\beta+\alpha^3+\alpha\beta^2+3\alpha\beta^2+\alpha^2\beta+\beta^3)$

$=-(8\alpha^3+32\alpha^2\beta+32\alpha\beta^2+8\beta^3)$

= LHS as required.

[2]

Alternatively

$\alpha+\beta=-\frac{p}{2}$

But $\alpha+\beta$ is a root.

$\therefore P\left(-\frac{p}{2}\right)=0$

$\left(-\frac{p}{2}\right)^3+p\left(-\frac{p}{2}\right)^2+q\left(-\frac{p}{2}\right)+r=0$

$-\frac{p^3}{8}+\frac{p^3}{4}+\left(-\frac{pq}{2}\right)+r=0$

$\frac{p^3}{8}+\left(-\frac{pq}{2}\right)+r=0$

$\therefore p^3+8r=4pq$ [3]

Question (6)

(a)

(i) $\cos x - \sqrt{3}\sin x = R\cos(\alpha + x)$

$R = \sqrt{1+3}$	$= 2$
$\tan\alpha = \frac{\sqrt{3}}{1} \Rightarrow$	$\alpha = \frac{\pi}{3}$

1

(ii) $2\cos\left(x+\frac{\pi}{3}\right) = 1$

$\therefore \cos\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$

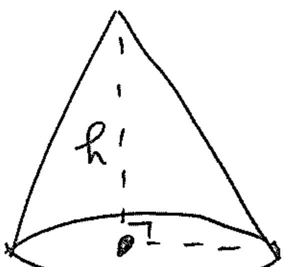
$x + \frac{\pi}{3} = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$

2 $\therefore x = 0, \frac{4\pi}{3}, 2\pi$

3

(b)

(i)



$r = \frac{h}{2}$

$V = \frac{\pi}{3}r^2 h = \frac{\pi}{3}\cdot\frac{h^2}{4}\cdot h$

$\therefore V = \frac{\pi h^3}{12}$ 1

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$= \frac{1}{(dV/dh)} \times \frac{dV}{dt}$

$= \frac{4}{\pi h^2} \times 20$

$= \frac{80}{\pi h^2}$ cm/s. 2

(ii) $\frac{\cancel{80}\ 10}{\pi \times \cancel{64}\ 8} = \frac{5}{4\pi}$ cm/s. 1

(0.398)

(iii) $\frac{dh}{dt} = \frac{80}{\pi h^2}$

$\therefore \frac{dt}{dh} = \left(\frac{\pi}{80}\right)h^2$

$t = \frac{\pi}{80}\int_0^8 h^2\,dh$ 1

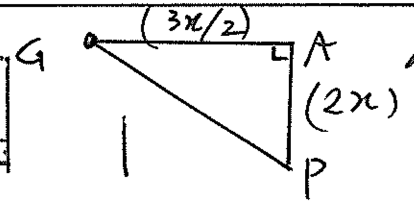
$= \left[\frac{\pi h^3}{240}\right]_0^8$

$= \frac{32\pi}{15}$ sec

$\doteqdot 6.7$ sec.

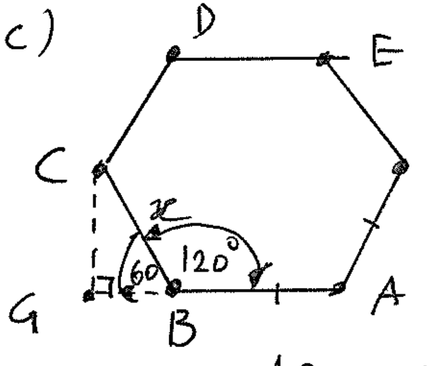
5

$\boxed{AG = \frac{x}{2} + x = 3x/2}$



1

(c)



4

$AB = x = CB$

Express $\left\{\begin{array}{l} BG \\ CG \end{array}\right.$ in terms of x

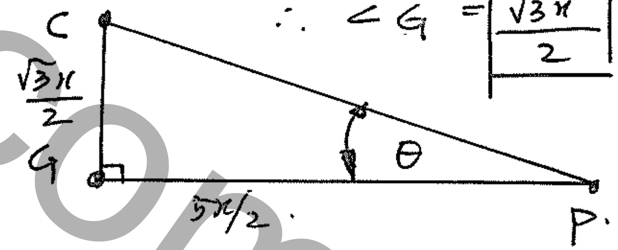
$\angle CBA = 120°$ (regular hex)

$\Longrightarrow CBG = 60°$

$\frac{BG}{x} = \cos 60° = \boxed{\frac{x}{2}}$ 1

$\frac{CG}{x} = \frac{\sqrt{3}}{2}$

$\therefore CG = \boxed{\frac{\sqrt{3}x}{2}}$



$GP = \sqrt{4x^2 + \frac{9x^2}{4}}$

$= \sqrt{\frac{25}{4}x^2} = \frac{5x}{2}$. 2

$\therefore \tan\theta = \frac{\sqrt{3}\cancel{x}}{\cancel{2}} \times \frac{\cancel{2}}{5\cancel{x}}$

$= \sqrt{3}/5$

2009 Mathematics Extension 1 Trial HSC: **Question 7** solutions

7. (a) Use mathematical induction to prove that $\cos(\pi n) = (-1)^n$, where n is a positive integer. **[2]**

> **Solution:** Test for $n = 1$:
>
> $$\begin{aligned} \text{L.H.S.} &= \cos\pi, & \text{R.H.S.} &= (-1)^1, \\ &= -1. & &= -1. \end{aligned}$$
>
> $\therefore$ True when $n = 1$.
>
> Now assume true when $n = k$, some particular integer,
>
> *i.e.* $\cos(\pi k) = (-1)^k$.
>
> Then test for $n = k + 1$, *i.e.* $\cos\big(\pi(k+1)\big) = (-1)^{k+1}$.
>
> $$\begin{aligned} \text{L.H.S.} &= \cos\big(\pi(k+1)\big), \\ &= \cos(\pi k + \pi), \\ &= \cos\pi k\cos\pi - \sin\pi k\sin\pi, \\ &= (-1)^k.(-1) - 0, \text{ using the assumption,} \\ &= (-1)^{k+1}, \\ &= \text{R.H.S.} \end{aligned}$$
>
> $\therefore$ True for all $n \geqslant 1$ by the principle of mathematical induction.

(b) (i) Find the largest possible domain of positive values for which $f(x) = x^2 - 5x + 13$ has an inverse. **[3]**

> **Solution:** $f'(x) = 2x - 5$,
>
> $2x - 5 = 0$ when $x = 5/2$.
>
> $\therefore$ Function is one-one if $x > 5/2$.

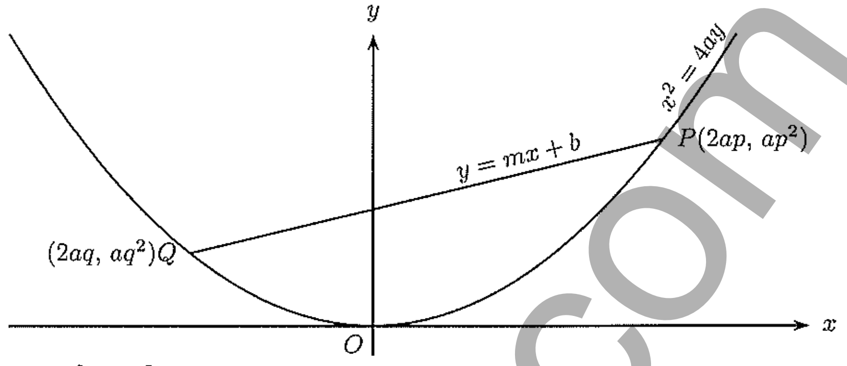
(ii) Find the equation of the inverse function, $f^{-1}(x)$.

> **Solution:** Put
>
> $$\begin{aligned} x &= y^2 - 5y + 13, \\ &= y^2 - 5y + \tfrac{25}{4} + 13 - \tfrac{25}{4}, \\ x - \tfrac{27}{4} &= (y - 5/2)^2, \\ y - 5/2 &= \frac{\pm\sqrt{4x - 27}}{2}, \\ y &= \frac{5 \pm \sqrt{4x-27}}{2}, \\ \textit{i.e. } f^{-1}(x) &= \frac{5 + \sqrt{4x-27}}{2}, \text{ taking the positive root as } f^{-1}(x) > 5/2. \end{aligned}$$

(c) The straight line $y = mx + b$ meets the parabola $x^2 = 4ay$ at the points $P(2ap,\ ap^2)$ and $Q(2aq,\ aq^2)$. 7

(i) Find the equation of the chord PQ and hence or otherwise show that $pq = -\frac{b}{a}$.

Solution:



$$\begin{aligned} y - ap^2 &= \frac{ap^2 - aq^2}{2ap - 2aq}(x - 2ap), \\ &= \frac{p+q}{2}(x - 2ap), \\ 2y - 2ap^2 &= (p+q)x - 2ap^2 - 2apq, \\ 2y &= (p+q)x - 2apq \text{ is the equation of } PQ. \end{aligned}$$

This is the same line as $y = mx + b$ so $b = -apq$ and thus $pq = -\dfrac{b}{a}$.

(ii) Prove that $p^2 + q^2 = 4m^2 + \dfrac{2b}{a}$.

Solution: $m = \dfrac{p+q}{2}$,

$$\begin{aligned} \therefore \text{ R.H.S.} &= 4\left(\frac{p+q}{2}\right)^2 + 2(-pq), \\ &= p^2 + 2pq + q^2 - 2pq, \\ &= p^2 + q^2, \\ &= \text{L.H.S.} \end{aligned}$$

(iii) Given that the equation of the normal to the parabola at P is $x + py = 2ap + ap^3$ and that N, the point of intersection of the normals at P and Q, has coördinates

$$\left[-apq(p+q),\ a(2 + p^2 + pq + q^2)\right],$$

express these coördinates in terms of a, m and b.

Solution: Now $-apq = b,\ p + q = 2m,\ p^2 + q^2 = 4m^2 + 2b/a$.

$$\begin{aligned} \therefore\ x_N = 2bm, \quad y_N &= a(2 + 4m^2 + 2b/a - b/a), \\ &= a(2 + 4m^2 + b/a). \end{aligned}$$

$\therefore\ N : [2bm,\ 2a + 4am^2 + b]$

(iv) Suppose that the chord PQ is free to move while maintaining a fixed gradient. Find the locus of N and show that this locus is a straight line.

Verify that this line is a normal to the parabola.

Solution: Method 1—

$$b = \frac{x}{2m},$$

$$y = \frac{x}{2m} + 2a + 4am^2 \text{ which is the locus of } N$$

and a straight line with a slope of $^1/_{2m}$.

Rewriting, $x - 2my = -4am - 8am^3$,
then let $p = -2m$ so that $x + py = 2ap + ap^3$
which is in the form of a normal to the parabola $x^2 = 4ay$.

Solution: Method 2—

$$b = \frac{x}{2m},$$

$$y = \frac{x}{2m} + 2a + 4am^2 \text{ which is the locus of } N$$

and a straight line with a slope of $\frac{1}{2m}$.

Where this locus of N meets the parabola $x^2 = 4ay$,

$$\begin{aligned}
x^2 &= 4a\left(\frac{x}{2m} + 2a + 4am^2\right), \\
mx^2 - 2ax - 8a^2m + 16a^2m^3 &= 0. \\
x &= \frac{2a \pm \sqrt{4a^2 + 4a^2\left(8m^2 + 16m^4\right)}}{m}, \\
&= \frac{a \pm a\sqrt{1 + 8m^2 + 16m^4}}{m}, \\
&= \tfrac{a}{m}\left(1 \pm \left(1 + 4m^2\right)\right), \\
&= \tfrac{a}{m}\left(2 + 4m^2\right) \text{ or } \tfrac{a}{m}\left(-4m^2\right).
\end{aligned}$$

In the limiting case when $x = -4am$, $p = q$ and $-4am = 2ap$,

$$\therefore\ p = -2m.$$

So the slope of the normal at this point is $\frac{1}{2m}$ which is the slope of the locus of N.