SYDNEY BOYS HIGH SCHOOL

MOORE PARK, SURRY HILLS

2004

YEAR 12

HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK # 3

Mathematics Extension 1

General Instructions

- Working time – 90 minutes.
- Reading Time – 5 minutes.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may not be awarded for messy or badly arranged work

Total Marks - 66

- Attempt *all* questions
- *All* questions are of equal value
- Return your answers in 3 booklets, one for each section. Each booklet must show your student number.

Examiner: *Mr R Dowdell*

Standard Integrals

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \; n \neq -1; \; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \; x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left\{x + \sqrt{x^2 - a^2}\right\}, \; |x| > |a|$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left\{x + \sqrt{x^2 + a^2}\right\}$$

NOTE: $\ln x = \log_e x$

Section A:

Question 1: (11 marks)

Marks

(a) Evaluate $\int_0^2 \frac{dx}{\sqrt{16-x^2}}$ 2

(b) Evaluate

(i) $\lim_{x \to 0} \frac{\sin 3x}{4x}$

(ii) $\lim_{x \to 0} \frac{\sin 3x}{\sin 7x}$ 3

(c) Use the substitution $u = \ln x$ to find $\int \frac{dx}{x\sqrt{1-(\ln x)^2}}$. 2

(d) Differentiate $\log_e(\sin^3 x)$, writing your answer in simplest form. 2

(e) Differentiate with respect to x, $(\tan^{-1} x)^2$. 2

Question 2: (11 marks)

Marks

(a) (i) Write down the domain and range of $y = \sin^{-1}(\sin x)$.

(ii) Draw a neat sketch of $y = \sin^{-1}(\sin x)$. 3

(b) Given that $y = \sin^{-1}(\sqrt{x})$, show that $\dfrac{dy}{dx} = \dfrac{1}{\sin 2y}$. 3

(c) Show that the derivative of $x \tan x - \ln(\sec x)$ is $x \sec^2 x$.

Hence, or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{4}} x \sec^2 x \, dx$. 3

(d) If $y = 10^x$, find $\dfrac{dy}{dx}$ when $x = 1$. 2

Section B:

Question 3: (11 marks) **START A NEW BOOKLET** Marks

(a) Consider the function $y = 4\sin\left(x + \frac{\pi}{6}\right), \ \frac{\pi}{3} \le x \le \frac{4\pi}{3}$.

(i) Find the inverse function of *y*, and write down its domain. 4

(ii) Sketch the inverse function of *y*.

(b) (i) On the same axes, draw the graphs of $y = \tan^{-1} x$ and $y = \cos^{-1} x$, showing the important features. Mark the point *P* where the curves intersect.

(ii) Show that, if $\tan^{-1} x = \cos^{-1} x$, then $x^4 + x^2 - 1 = 0$. Hence, find the coordinates of *P*, correct to 2 decimal places. 5

(c) Show that $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{4}$ 2

Question 4: (11 marks)

Marks

(a) (i) Draw a neat sketch of $y = \cos^{-1} x$. State its domain and range.

(ii) Shade the area bounded by $y = \cos^{-1} x$ and the x and y axes on your diagram. 4

(iii) Calculate the area of the region specified in (ii).

(b) Differentiate $y = \log_e\left(\dfrac{2x}{(x-1)^2}\right)$. Write your answer in simplest form. 2

(c) The rate of change of temperature T^o, of an object is given by the equation $\dfrac{dT}{dt} = k(T-16)$ degrees per minute, k a constant.

(i) Show that the function $T = 16 + Pe^{kt}$, where P is a constant and t the time in minutes, satisfies the equation.

(ii) If initially $T = 0$ and after 10 minutes $T = 12$, find the values of P and k. 5

(iii) Find the temperature of the object after 15 minutes.

(iv) Sketch the graph of T as a function of t and describe its behaviour as t continues to increase.

Section C:

Question 5: (11 marks) **START A NEW BOOKLET** Marks

(a) It is known that $\ln x + \sin x = 0$ has a root close to $x = 0 \cdot 5$. Use one application of Newton's method to obtain a better approximation (to 2 decimal places). 2

(b) The acceleration of a particle P is given by the equation $\ddot{x} = 8x(x^2 + 1)$ ms^{-2}, where x is the displacement of P from the origin in metres after t seconds, with movement being in a straight line.

Initially the particle is projected from the origin with a velocity of 2 ms^{-1}.

(i) Show that the velocity of the particle can be expressed as $v = 2(x^2 + 1)$. 6

(ii) Hence, show that the equation describing the displacement of the particle at time t is given by $x = \tan 2t$.

(iii) Determine the velocity of the particle at time $\dfrac{\pi}{8}$ seconds.

(c) The arc of the curve $y = \sin^{-1} x$ between $x = 0$ and $x = 1$ is rotated about the x axis. Use Simpson's Rule with three function values to estimate the volume of the solid formed. 3

Question 6: (11 marks)

Marks

(a) The velocity v ms^{-2} of a particle moving in simple harmonic motion along the x axis is given by the expression $v^2 = 28 + 24x - 4x^2$.

(i) Between which two points is the particle oscillating?

(ii) What is the amplitude of the motion?

(iii) Find the acceleration in terms of x. 6

(iv) Find the period of the oscillation.

(v) If the particle starts from the point furthest to the right, find the displacement in terms of t.

(b) A stone is thrown from the top of a vertical cliff over the water of a lake. The height of the cliff is 8 metres above the level of the water, the initial speed of the stone is 10 ms^{-1} and the angle of projection is $\theta = \tan^{-1}\left(\frac{3}{4}\right)$ above the horizontal.

The equations of motion of the stone, with air resistance neglected, are $\ddot{x} = 0$ and $\ddot{y} = -g$.

(i) By taking the origin O as the base of the cliff, show that the horizontal and vertical components of the stone's displacement from the origin after t seconds are given by $x = 8t$ and $y = -\frac{1}{2}gt^2 + 6t + 8$. 5

(ii) Hence, or otherwise, calculate the time which elapses before the stone hits the lake and find the horizontal distance of the point of contact from the base of the cliff. (Assume g = 10 ms^{-2}.)

End of Paper