Ext. 1 (3U) – Solutions TRIAL 2003.

Question 1

(a) $\frac{d}{dx}\left(x\tan^{-1}2x\right) = \tan^{-1}2x + x \times \frac{2}{1+4x^2}$

$= \tan^{-1}2x + \frac{2x}{1+4x^2}$ (2)

(b) $x = t^2, \quad y = t^3 + t$.

$t = \pm\sqrt{x}$; $y = x\sqrt{x} + \sqrt{x}$ or $y = -x\sqrt{x} - \sqrt{x}$ (2)

(c) $\sin x = \frac{1}{2}$.

$x = n\pi + (-1)^n \frac{\pi}{6}$ (2)

(d) $A(5,4)$ $B(x,y)$ $2:3$ $P(-1,3)$

$\frac{2x + 3\times5}{5} = -1$ $\qquad$ $\frac{2y + 3\times4}{5} = 3$

$2x + 15 = -5$ $\qquad$ $2y + 12 = 15$

$x = -10$ $\qquad$ $y = 1\frac{1}{2}$. (2)

(e) $\lim_{x\to0} \frac{\sin 3x}{4x} = \lim_{x\to0} \frac{3}{4}\frac{\sin 3x}{3x}$

$= \frac{3}{4}$. (2)

(f) $\int_0^1 \frac{1}{\sqrt{4+x^2}}\,dx = \left[\ln\left(x + \sqrt{4+x^2}\right)\right]_0^1$

$= \ln(1+\sqrt{5}) - \ln 2$ (2)

Question 2

(a) $x + y - 4 = 0$ $\qquad$ $y = 2x + 1$

$m = -1$ $\qquad$ $m = 2$.

$\tan\theta = \frac{2-(-1)}{1+2\times(-1)}$

$= -3$

$\theta = 180^\circ - 72^\circ$

$= 108^\circ$. (2)

(b) $\frac{2x+3}{x-4} \le 1$ $\qquad$ $x \ne 4$.

$(2x+3)(x-4) \le (x-4)^2$.

$(x-4)\left[(x-4) - (2x+3)\right] \ge 0$

$(x-4)(-x-7) \ge 0$

$-7 \le x < 4$ $\quad (x \ne 4)$ (3)

(c) $\int_0^1 x\sqrt{2-x}\,dx = \int_2^1 (2-u)\sqrt{u}\,(-du)$

$u = 2-x$
$x = 2-u$
$x=0,\ u=2$
$x=1,\ u=1$
$du = -dx$

$= \int_1^2 \left(2u^{\frac{1}{2}} - u^{\frac{3}{2}}\right)du$

$= \left[2\times\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]_1^2$

$= \left(\frac{4}{3}\times2\sqrt{2} - \frac{2}{5}\times4\sqrt{2}\right) - \left(\frac{4}{3} - \frac{2}{5}\right)$

$= 8\sqrt{2}\left(\frac{1}{3} - \frac{1}{5}\right) - \frac{14}{15}$

$= \frac{16\sqrt{2} - 14}{15}$ (4)

(d) (i) $y = \frac{\pi}{2} - \sin^{-1}\frac{x}{2}$

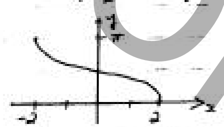
Domain: $-1 \le \frac{x}{2} \le 1$ $\therefore$ $-2 \le x \le 2$

Range: $-\frac{\pi}{2} \le \sin^{-1}\frac{x}{2} \le \frac{\pi}{2}$

$\frac{\pi}{2} \ge -\sin^{-1}\frac{x}{2} \ge -\frac{\pi}{2}$

$\pi \ge \frac{\pi}{2} - \sin^{-1}\frac{x}{2} \ge 0$ (2)

(ii)



(1)

Question 3.

(a) $\cos\frac{7\pi}{12} = \cos 105^\circ = \cos(45^\circ+60^\circ)$

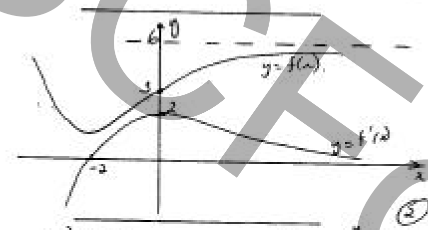
$= \cos 45^\circ\cos 60^\circ - \sin 45^\circ \sin 60^\circ$

$= \frac{1}{\sqrt{2}} \times \frac{1}{2} - \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}$

$= \frac{1-\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{2}-\sqrt{6}}{4}$ (3)

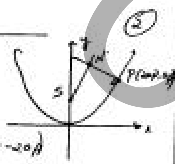
(b)



(2)

(c) $x^2 = 4ay$

$\frac{dy}{dx} = \frac{2x}{4a}$



(i) at $x = 2ap$, $\frac{dy}{dx} = \frac{2 \times 2ap}{4a} = p$

Eqn of normal: $y - ap^2 = -\frac{1}{p}(x - 2ap)$

$py - ap^3 = -x + 2ap$

$x + py = 2ap + ap^3$ (A) (2)

(ii) SN: $y = px + a$ (1)

(iii) Sub $y = px + a$ into (A)

$x + p(px + a) = 2ap + ap^3$

$x + p^2x + ap = 2ap + ap^3$

$x(1+p^2) = ap(1+p^2)$

$x = ap$, $y = p(ap) + a$

N: $x = ap$, $y = ap^2 + a$ (2)

(iv) $p = \frac{x}{a}$ ∴ $y = a \times \frac{x^2}{a^2} + a$

$y = \frac{x^2}{a} + a$ which is a parabola

$ay = x^2 + a^2$

$x^2 = a(y - a)$

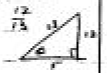
Vertex is $(0, a)$. (2)

Question 4.

(a) $\cos\left[2\sin^{-1}\left(\frac{12}{13}\right)\right]$

Let $\theta = \sin^{-1}\frac{12}{13}$

$\sin\theta = \frac{12}{13}$



$= \cos 2\theta$

$= 1 - 2\sin^2\theta$

$= 1 - 2 \times \frac{144}{169}$

$= -\frac{119}{169}$ (2)

(b)(i) Let $f(x) = x - \tan^{-1} 3x$

$f(1) = 1 - \tan^{-1} 3 = -0.249$

$f(2) = 2 - \tan^{-1} 6 = 0.594$

Since $f(1)$, $f(2)$ have opposite signs & $f(x)$ is continuous there is a root between $x = 1$, $x = 2$ (1)

(ii) $f'(x) = 1 - \frac{3}{1+9x^2}$

$f'(1.5) = 1 - \frac{3}{1+9 \times 1.5^2} = 0.8588$

$f(1.5) = 1.5 - \tan^{-1} 4.5 = 0.1479$.

Second approx'n $= 1.5 - \frac{0.1479}{0.8588}$

$= 1.33$ (2 dp) (2)

(c) $v = 4x + 1$

$accel = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\frac{1}{2}(4x+1)^2$

$= 4(4x+1)$

when $x = 5$, accel^n = 84 m/s². (2)

(d) $T = 22 + Ae^{kt}$

(i) when $t = 0$, $T = 85$ ∴ $85 = 22 + Ae^0$ ∴ $A = 63$ (1)

(ii) when $t = 3$, $T = 70$ ∴ $70 = 22 + 63e^{3k}$

$e^{3k} = \frac{48}{63}$

$k = \frac{1}{3}\ln\left(\frac{48}{63}\right)$

$= -0.091$ (3 dp) (2)

(iii) when $t = 12$, $T = 22 + 63e^{(-0.091 \times 12)}$

$= 43$ degrees (nearest degree) (2)

Question 5

$2x^3 - 3x^2 - 3x + 2 = 0$ $\quad \frac{\alpha}{r}, \alpha, \alpha r$

(i) Sum of roots $= \frac{3}{2}$ (1)

(ii) Product of roots $= -\frac{2}{2} = -1$. (1)

(iii) $\frac{\alpha}{r} \times \alpha \times \alpha r = -1$

$\alpha^3 = -1$

$\alpha = -1$

$\frac{-1}{r} + (-1) + (-1)r = \frac{3}{2}$

$-\frac{1}{r} - r = \frac{5}{2}$

$-2 - 2r^2 = 5r$

$2r^2 + 5r + 2 = 0$

$(2r+1)(r+2) = 0$

$r = -\frac{1}{2}, -2$ (2)

(i) $AP = x \tan 30^\circ$ $\quad BP = x \tan 60^\circ$ (1)

(ii) $x^2 \tan^2 60^\circ - x^2 \tan^2 30^\circ = 1000^2$

$3x^2 - \frac{x^2}{3} = 1000000$

$\frac{8x^2}{3} = 1000000$

$x^2 = \frac{3000000}{8}$

$x = 610$ m (nearest 10 m) (3)

Prove $9^{n+2} - 4^n$ is divisible by 5.

when $n=1$, $9^{n+2} - 4^n = 9^3 - 4 = 725$

$\therefore$ true for $n = 1$.

Assume true for $n=k$, i.e. assume $9^{k+2} - 4^k = 5N$ (N integer)

then $n = k+1$, $9^{n+2} - 4^n = 9^{k+1+2} - 4^{k+1}$

$= 9 \times 9^{k+2} - 4 \times 4^k$

$= 9(5N + 4^k) - 4 \times 4^k$ (by assumption)

$= 45N + 5 \times 4^k$

$= 5(9N + 4^k)$ - divisible by 5

$\therefore$ if true for $n=k$, then true for $n = k+1$.

Since true for $n=1$, then true for $n = 2, 3, \ldots$ (4)

Question 6

(a) (i) $f(x) = 4 - \sqrt{x-1}$.

Domain: $x - 1 \geq 0$ i.e. $x \geq 1$

Range: $f(x) \leq 4$. (2)

(ii) $y = 4 - \sqrt{x-1}$

Inverse: $x = 4 - \sqrt{y-1}$

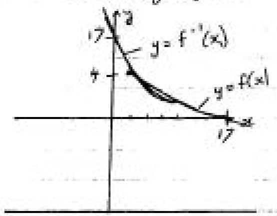
$x - 4 = -\sqrt{y-1}$

$(x-4)^2 = y - 1$

$y = (x-4)^2 + 1$

Domain: $x \leq 4$, Range, $y \geq 1$. (3)

(iii)



(2)

(b) $V = \frac{1}{3}\pi r^2 h$ $\quad \frac{r}{h} = \frac{8}{10}$

$= \frac{1}{3}\pi\left(\frac{4}{5}h\right)^2 h$ $\quad r = \frac{4}{5}h$

$= \frac{16}{75}\pi h^3$ $\quad \frac{dr}{dt} = \frac{4}{5}\frac{dh}{dt}$

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$35 = \frac{16\pi}{25}h^2 \frac{dh}{dt}$

when $h = 0.65$,

$35 = \frac{16\pi}{25} \times 0.65^2 \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{35 \times 25}{16\pi \times 0.65^2}$

$= 41.2$

$\frac{dr}{dt} = \frac{4}{5}\frac{dh}{dt}$

$= \frac{4}{5} \times 41.2$

$= 32.96$

Radius is decreasing at 32.96 m/s. (5)

Question 5

(a) $2x^3 - 3x^2 - 3x + 2 = 0$ $\quad \frac{\alpha}{r}, \alpha, \alpha r$.

(i) Sum of roots $= \frac{3}{2}$ (1)

(ii) Product of roots $= -\frac{2}{2} = -1$. (1)

(iii) $\frac{\alpha}{r} \times \alpha \times \alpha r = -1$

$\alpha^3 = -1$

$\alpha = -1$

$\frac{-1}{r} + (-1) + (-1)r = \frac{3}{2}$

$-\frac{1}{r} - r = \frac{5}{2}$

$-2 - 2r^2 = 5r$

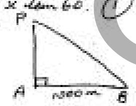
$2r^2 + 5r + 2 = 0$

$(2r+1)(r+2) = 0$

$r = -\frac{1}{2}, -2$ (2)

(b) (i) $AP = x \tan 30^\circ$ $\quad BP = x \tan 60^\circ$ (1)

(ii) $x^2 \tan^2 60^\circ - x^2 \tan^2 30^\circ = 1000^2$



$3x^2 - \frac{x^2}{3} = 1000000$

$\frac{8x^2}{3} = 1000000$

$x^2 = \frac{3000000}{8}$

$x = 610$ m (nearest 10 m) (3)

(c) Prove $9^{n+2} - 4^n$ is divisible by 5.

when $n=1$, $9^{n+2} - 4^n = 9^3 - 4 = 725$

$\therefore$ True for $n=1$.

Assume true for $n=k$, i.e. assume $9^{k+2} - 4^k = 5N$ (N integer)

when $n=k+1$, $9^{n+2} - 4^n = 9^{k+1+2} - 4^{k+1}$

$= 9 \times 9^{k+2} - 4 \times 4^k$

$= 9(5N + 4^k) - 4 \times 4^k$ (by assumption)

$= 45N + 5 \times 4^k$

$= 5(9N + 4^k)$ - divisible by 5

$\therefore$ if true for $n=k$, then true for $n=k+1$.

Since true for $n=1$, then true for $n=2, 3, \ldots$ (4)

Question 6

(a) (i) $f(x) = 4 - \sqrt{x-1}$.

Domain: $x - 1 \geq 0$ i.e. $x \geq 1$

Range: $f(x) \leq 4$. (2)

(ii) $y = 4 - \sqrt{x-1}$

Inverse: $x = 4 - \sqrt{y-1}$

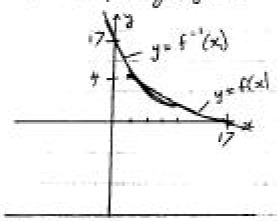
$x - 4 = -\sqrt{y-1}$

$(x-4)^2 = y - 1$

$y = (x-4)^2 + 1$

Domain: $x \leq 4$, Range, $y \geq 1$. (3)

(iii)



(2)

(b) $V = \frac{1}{3}\pi r^2 h$. $\quad \frac{r}{h} = \frac{8}{10}$

$= \frac{1}{3}\pi\left(\frac{4}{5}h\right)^2 h$ $\quad r = \frac{4}{5}h$

$= \frac{16}{75}\pi h^3$ $\quad \frac{dr}{dt} = \frac{4}{5}\frac{dh}{dt}$

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$35 = \frac{16\pi}{25} h^2 \frac{dh}{dt}$

when $h = 0.65$

$35 = \frac{16\pi}{25} \times 0.65^2 \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{35 \times 25}{16\pi \times 0.65^2}$

$= 41.2$

$\frac{dr}{dt} = \frac{4}{5}\frac{dh}{dt}$

$= \frac{4}{5} \times 41.2$

$= 32.96$

Radius is decreasing at 32.96 m/s. (3)

Question 1

$$\cos x + \frac{1}{\sqrt{2}}\sin x = -1$$

Let $t = \tan\frac{x}{2}$: $\quad \frac{1-t^2}{1+t^2} + \frac{1}{\sqrt{2}} \times \frac{2t}{1+t^2} = -1$

$$1 - t^2 + \sqrt{2}t = -1 - t^2$$

$$\sqrt{2}t = -2.$$

$$t = -\sqrt{2}.$$

$\tan\frac{x}{2} = -\sqrt{2} \quad \therefore \frac{x}{2} = 180^\circ - 55^\circ$

$$x = 250^\circ.$$

Test $x = 180^\circ$: $\cos x + \frac{1}{\sqrt{2}}\sin x = \cos(180) + \frac{1}{\sqrt{2}}\sin 180$

$$= -1 - \frac{1}{\sqrt{2}} \times 0$$

$$= -1.$$

$\therefore$ Solutions are 180, 250° (nearest degree)

4

$$V = \pi\int_0^{\frac{\pi}{2}} y^2\,dx$$

$$= \pi\int_0^{\frac{\pi}{2}} \cos^2 2x\,dx$$

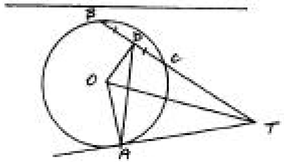
$$= \pi\int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 4x)\,dx$$

$$= \frac{\pi}{2}\left[x + \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}\left[\left(\frac{\pi}{2} + \frac{1}{4}\sin 2\pi\right) - (0 + 0)\right]$$

Volume $= \frac{\pi^2}{4}$ unit^3

4



d) $\angle OAT = 90^\circ$ (angle between tangent & radius to pt. of contact)

$\angle ODT = 90^\circ$ (line from centre of circle to midpoint of a chord bisects the chord)

Since Opp. angles of AODT are supplementary,

AODT is a cyclic quad

When a circle is drawn through AODT,

$\angle AOT$ and $\angle ADT$ are angles at circumference standing on same chord AT

3

$\therefore \angle AOT = \angle ADT$.

1