FORM VI MATHEMATICS EXTENSION 1

Time allowed: 2 hours (plus 5 minutes reading) **Exam date:** 15th August 2002

Instructions:

All questions may be attempted.
All questions are of equal value.
Part marks are shown in boxes in the right margin.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection:

Each question will be collected separately.
Start each question in a new answer booklet.
If you use a second booklet for a question, place it inside the first. Don't staple.
Write your candidate number on each answer booklet.

Checklist:

SGS Examination booklets required — seven 4-page books per boy.
120 candidates.

QUESTION ONE (Start a new answer booklet)

Marks

(a) Evaluate $\sum_{n=1}^{4} n^2$. (Show your working.) **1**

(b) Solve the inequation $\dfrac{x}{x-3} > 2$. **3**

(c) (i) Differentiate $y = e^{7x}$. **1**

(ii) Hence find the equation of the tangent to $y = e^{7x}$ at $x = 1$. **3**

(d) Simplify the expression $\dfrac{{}^{n}C_2}{{}^{n}C_1}$. **2**

(e) Write down the general solution of the equation $\cos 3x = \cos \frac{\pi}{5}$. **2**

QUESTION TWO (Start a new answer booklet)

(a) The radius of a circular oil slick is increasing at 0·1m/s.

Marks

(i) Show that the rate of increase of its area is given by $\dfrac{dA}{dt} = 0{\cdot}2\pi r$. **2**

(ii) What is the radius when the area is increasing at 2π m^2/s? **1**

(b) Find the volume generated when the region between $y = \sec 3x$ and the x-axis, from $x = -\frac{\pi}{12}$ to $x = \frac{\pi}{12}$, is rotated about the x-axis. **3**

(c) Prove the identity $\sin 2\theta(\tan\theta + \cot\theta) = 2$. **3**

(d) A particle is moving with acceleration $\ddot{x} = -9x$ and it is initially stationary at $x = 4$.

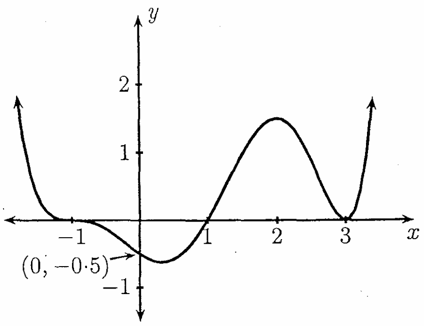
(i) Find v^2 as a function of x. **2**

(ii) What is the particle's maximum speed? **1**

QUESTION THREE (Start a new answer booklet)

Marks

(a) **3**



Write down a possible equation $y = P(x)$ for the polynomial function sketched above. (Do not use calculus.)

(b) Find the term independent of x in the expansion of $\left(x^2 + \dfrac{3}{x}\right)^{12}$. **3**

(c) A particle's displacement is $x = 5 - 3\sin(2t + \frac{\pi}{4})$, in units of centimetres and seconds.

(i) In what interval is the particle moving? **1**

(ii) Write down the period of the motion. **1**

(iii) Find the first two times after time zero when the particle is closest to the origin. **2**

(d) (i) Write down the expansion of $(1 + 2x)^n$. **1**

(ii) Hence prove the identity **1**

$$3^n = {}^nC_0 + 2 \times {}^nC_1 + 2^2 \times {}^nC_2 + \cdots + 2^n \times {}^nC_n.$$

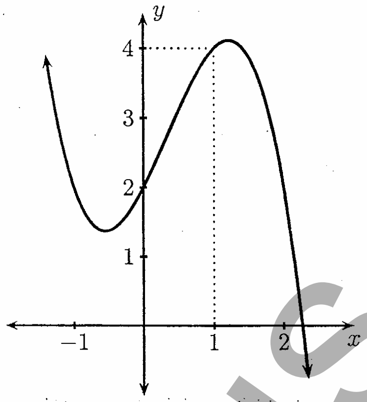
QUESTION FOUR (Start a new answer booklet)

Marks

(a) Use the substitution $u = 1 - x$ to evaluate **4**

$$\int_0^1 x(1-x)^7 \, dx.$$

(b)



A pupil is using Newton's method to calculate an approximate value for the single zero of the polynomial $y = x^2 + 2x + 2 - x^3$. The polynomial is sketched above.

(i) Use Newton's method with $x_0 = 2$ to approximate the zero correct to two decimal places. **3**

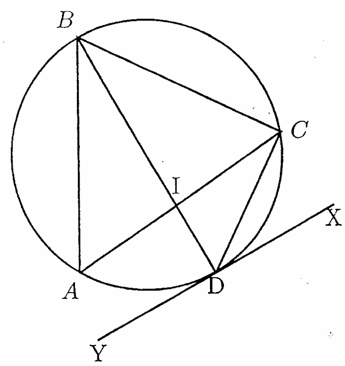
(ii) Quickly copy the graph above and use it to show why $x_0 = 1$ would be a bad initial approximation to the root (no calculations are required). **1**

(c) (i) Show that $y = \sqrt{3}\sin x + \cos x$ may be expressed in the form $y = 2\sin(x + \frac{\pi}{6})$. **2**

(ii) Hence sketch the curve $y = \sqrt{3}\sin x + \cos x$, for $0 \le x \le 2\pi$. Do not use calculus but do mark x- and y-intercepts. **2**

QUESTION FIVE (Start a new answer booklet)

(a)



In the circle above, $\angle ABD + \angle BCA = 90^\circ$ and XY is a tangent to the circle at D. The chords AC and BD intersect at I.

Marks

(i) Prove that $\angle BCD = 90^\circ$ and hence that BD is a diameter of the circle. **2**

(ii) Prove that if $\triangle ABC$ is isosceles with $AB = BC$, then $AC || XY$. **2**

(b) Harry is investing a certain amount each year for 20 years in a superannuation fund, which pays 6% per annum compound interest. Harry pays a yearly contribution $\$M$ at the start of each year.

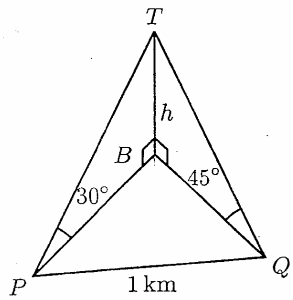
(i) Show that after n years Harry's investment amounts to **2**

$$M \times 1{\cdot}06 \; + \; M \times 1{\cdot}06^2 \; + \; \cdots \; + \; M \times 1{\cdot}06^n.$$

(ii) By summing this geometric series, find M if the fund is to reach \$500 000 at the end of the 20 years. **2**

QUESTION FIVE (Continued)

(c)



The angle of elevation from a boat at P to a point T at the top of a vertical cliff is measured to be 30°. The boat sails 1 km to a second point Q, from which the angle of elevation to T is measured to be 45°. Let B be the point at the base of the cliff directly below T and let $h = BT$ be the height of the cliff in metres. The bearings of B from P and Q are 50° and 310° respectively.

(i) Show that $\angle PBQ = 100°$. **[1]**

(ii) Find expressions for PB and QB in terms of h. **[1]**

(iii) Hence show that **[1]**

$$h^2 = \frac{1000^2}{\cot^2 30° + \cot^2 45° - 2\cot 30° \cot 45° \cos 100°}$$

(iv) Use a calculator to find the height of the cliff, correct to the nearest metre. **[1]**

QUESTION SIX (Start a new answer booklet)

Marks

(a) (i) Show that the tangent to the parabola $x^2 = 8y$ at the point $P(4p, 2p^2)$ has equation $y = px - 2p^2$. **[2]**

(ii) By substituting the point $A(3, -2)$ into this equation of the tangent, or otherwise, find the points of contact of the tangent to the parabola from A. **[2]**

(b) (i) A particle moves along a path described by $y = \frac{1}{4}x^2$. Show that the perpendicular distance from any point on its path to the line $3x - 4y + 4 = 0$ is **[1]**

$\frac{1}{5}|x^2 - 3x - 4|$.

(ii) Sketch the curve $y = \frac{1}{5}|x^2 - 3x - 4|$, showing any intercepts and the vertex. **[2]**

(iii) Use your graph in part (ii) to show that the distance from the path to the line is $1\frac{1}{4}$ units on exactly three occasions. **[1]**

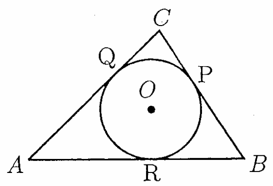
(c) Use mathematical induction to prove that for all integers $n > 0$, **[4]**

$$1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \cdots + n(n+1)^2 = \tfrac{1}{12}n(n+1)(n+2)(3n+5).$$

QUESTION SEVEN (Start a new answer booklet)

Marks

(a) **3**



The incircle of a triangle is the circle inscribed to touch the triangle at exactly three points, as in the diagram above. In the diagram, O is the centre of the incircle, and P, Q and R are the points where the incircle touches the triangle. Let r be the radius of the incircle and p be the perimeter of the triangle. Prove that

$$\text{area } \triangle ABC = \frac{1}{2}rp.$$

(b) (i) By differentiating, or otherwise, prove that for $x > -1$, **3**

$$\tan^{-1} x + \tan^{-1} \frac{1-x}{1+x} = \frac{\pi}{4}.$$

(ii) Hence, or otherwise, calculate $\tan^{-1}(\sqrt{2}-1)$. **1**

(c)

An artist has been commissioned to draw a piece of modern art, using his pack of 30 long sharp coloured pencils.
First he draws a square. Then he draws a vertical line that divides the area of the square in the ratio $2:1$. Then he draws further vertical lines that divide the areas of each region in the ratio $2:1$ (see the diagram). The artist repeats this process eleven times, resulting in $2^{11} = 2048$ thin rectangles.
The artist then begins to colour in the rectangles, using the same colour for rectangles of the same size, but different colours for rectangles of different sizes.

(i) Let the number of rectangles of size $\left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{n-k}$ be written $r(n,k)$. Show that **2**

$$r(n,k) = r(n-1,k-1) + r(n-1,k).$$

(ii) Show that he has enough colours to colour the entire square, and find what area is coloured by the pencil that he uses the most of. **3**

BDD