

2001 3 UNIT TRIAL MARKING SCHEME

Question 1

(a) $x = \frac{3 \times 4 + 1 \times (-5)}{3+1}$

$= \frac{7}{4}$

$y = \frac{3 \times (-3) + 1 \times 6}{4}$

$= -\frac{3}{4}$

the point is $(\frac{7}{4}, -\frac{3}{4})$

(b) $m_1 = -\frac{1}{2}, m_2 = \frac{1}{3}$

let θ be the acute angle

$\tan \theta = \left| \frac{-\frac{1}{2} - \frac{1}{3}}{1 + (-\frac{1}{2})(\frac{1}{3})} \right|$

$\theta = 45^\circ$

(c) (i) the angle at the centre is equal to twice the angle at the circumference when they are subtended by the same arc.

(ii) $\angle OBC = 60^\circ$ (alternate angles, $AO \parallel BC$)

$x = 90^\circ$ (angle sum of $\triangle ABC$)

(d) $P(x) = x^3 - x^2 - 10x - 8$

(i) $P(-1) = -1 - 1 + 10 - 8 = 0$

so $x = -1$ is a zero of $P(x)$

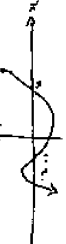
(ii) $(x+1)$ is a factor of $P(x)$

$$\begin{array}{r} x^3 - x^2 - 10x - 8 \\ x+1 \overline{) x^3 - x^2 - 10x - 8} \\ \underline{x^3 + x^2 + x + 1} \\ -2x^2 - 11x - 9 \\ \underline{-2x^2 - 2x - 2} \\ -9x - 7 \end{array}$$

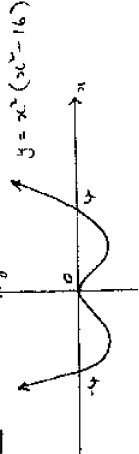
$$\begin{array}{r} -9x - 10x \\ \underline{-9x - 9} \\ -8x - 1 \end{array}$$

$P(x) = (x+1)(x^2 - 2x - 8)$
 $= (x+1)(x-4)(x+2)$

(iii) $P(x) \leq 0$



Question 2



(b) (i) $x^2 + 4x + 5 = (x+2)^2 + 1$

(ii) $\int \frac{dx}{x^2 + 4x + 5} = \int \frac{dx}{1 + (x+2)^2}$

$= \tan^{-1}(x+2) + C$

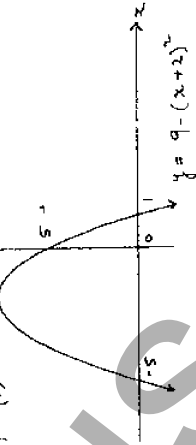
(c) $\cos 2x = \cos x$

$2x = 2n\pi \pm x$

$3x = 2n\pi$ or $x = 2n\pi$

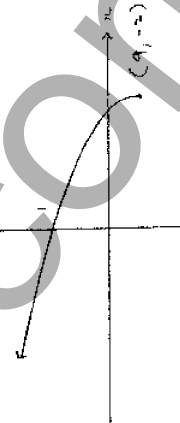
$x = \frac{2n\pi}{3}$ for any integer n

(d) (i) $(2, 5)$



(ii) $x \geq -2$

(iii)



QUESTION 3

$$(a) \lim_{x \rightarrow 0} \frac{\sin 4x}{\tan 3x} = \lim_{x \rightarrow 0} \frac{\sin 4x}{4x} \times \lim_{x \rightarrow 0} \frac{4x}{\tan 3x} = 1 \times 1 = 1$$

$$(b) (x + \frac{1}{x})^9$$

$$T_x = {}^9C_r x^r (\frac{1}{x})^{9-r}$$

$$= {}^9C_r x^r x^{-(9-r)} = {}^9C_r x^{2r-9}$$

for the term independent of x
 $3x - 18 = 0$
 $x = 6$

Hence the term is ${}^9C_6 = 84$

$$(c) \frac{dv}{dt} = 72$$

$$v = \frac{4}{3} \pi r^3, \quad S = 4\pi r^2$$

$$\frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt}, \quad \frac{dS}{dt} = 8\pi r$$

$$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt} = \frac{dS}{dr} \times \frac{dv}{dr}$$

$$= \frac{8\pi r \times 72}{4\pi r^2}$$

$$= \frac{2 \times 72}{r}$$

$$\text{when } r = 12, \quad \frac{dS}{dt} = 12 \text{ m}^2/\text{s}$$

(d) (i) Consider $f(x) = \sin x - x + \frac{1}{2}$

$$f(0.5) > 0$$

$$f(1.8) < 0$$

so, there is a root between $x = 0.5$ and $x = 1.8$

$$(ii) f'(x) = \cos x - 1$$

$$x = x_1 = \frac{f(x_1)}{f'(x_1)}$$

$$= 1.2 = \frac{f(1.2)}{f'(1.2)}$$

$$= 1.56 \quad (2 \text{ decimal places})$$

QUESTION 4

$$(a) 3 \sin x + \sqrt{3} \cos x = R \sin(x + \alpha)$$

$$= R \sin x \cos \alpha + R \cos x \sin \alpha$$

$$R \sin \alpha = \sqrt{3}$$

$$R \cos \alpha = 3$$

$$\tan \alpha = \frac{\sqrt{3}}{3}$$

$$\alpha = \frac{\pi}{6}$$

$$R = \sqrt{3^2 + (\sqrt{3})^2} = 2\sqrt{3}$$

$$3 \sin x + \sqrt{3} \cos x = 2\sqrt{3} \sin(x + \frac{\pi}{6})$$

$$(b) (i) \angle BDC = \alpha \quad (\text{base angles of isosceles } \Delta)$$

$$\angle DCR = 2\alpha \quad (\text{exterior angle of } \Delta BCD)$$

$$(ii) \angle BAD = 2\alpha \quad (\text{exterior angle of cyclic quad. } ABCD)$$

$$\therefore \angle OAD = \alpha \quad (OA \text{ bisects } \angle BAD)$$

(iii) $OA \perp AT$ (radius is perpendicular to the tangent at the point of contact)

$$\text{so, } \angle TAD = 90^\circ - \alpha$$

$$\angle ABD = \angle TAD \quad (\text{alternate segment theorem})$$

$$\text{so, } \angle ABC = (90^\circ - \alpha) + \alpha = 90^\circ$$

$$(c) (i) \text{ In } \Delta LMP: \tan 20^\circ = \frac{LM}{PM}$$

$$PM = 50 \cot 20^\circ \text{ metres}$$

$$(ii) PQ^2 = PM^2 + QM^2 = 2 \cdot PM \cdot QM \cdot \cos 60^\circ \quad (\text{cosine rule})$$

$$= 50^2 \cot^2 20^\circ + 50^2 \cot^2 12^\circ - 2 \cdot 50 \cot 20^\circ \cdot 50 \cot 12^\circ \cdot \cos 65^\circ$$

$$\text{so, } PQ = 50 \sqrt{\cot^2 20^\circ + \cot^2 12^\circ - 2 \cot 20^\circ \cot 12^\circ \cos 65^\circ}$$

$$(iii) \text{ Speed} = \frac{PQ}{10 \times 60}$$

$$= 0.36 \text{ m/s} \quad (2 \text{ sig. fig.})$$

QUESTIONS

2) (i) $\frac{d}{dx} (x \cos^{-1} x - \sqrt{1-x^2})$
 $= \cos^{-1} x - \frac{x}{\sqrt{1-x^2}} - \frac{-\frac{1}{2} \cdot 2x}{\sqrt{1-x^2}}$
 $= \cos^{-1} x$

(ii) $\int \cos^{-1} x \, dx = [x \cos^{-1} x - \sqrt{1-x^2}]_0^1$
 $= 1$

(b) $u = 1-x \Rightarrow x = 1-u$
 $du = -dx$

when $x = -3 \Rightarrow u = 4$

when $x = 0 \Rightarrow u = 1$

$I = \int_4^1 \frac{1-u}{\sqrt{u}} - du$

$= \int_1^4 u^{-1/2} - u^{1/2} du$

$= [2u^{1/2} - \frac{2}{3} u^{3/2}]_1^4$

$= (4 - \frac{2}{3} \cdot 8) - (\frac{2}{3} - \frac{2}{3})$
 $= -\frac{8}{3}$

(c) for $(1+x)^{2m}$ the coefficient of x^n is $\binom{2m}{n}$

$(1+x)^{2m} = (1+x)^m (1+x)^m$
 $= \left[\binom{m}{0} + \binom{m}{1}x + \binom{m}{2}x^2 + \dots + \binom{m}{n}x^n + \dots + \binom{m}{m}x^m \right] \left[\binom{m}{0} + \binom{m}{1}x + \binom{m}{2}x^2 + \dots + \binom{m}{n}x^n + \dots + \binom{m}{m}x^m \right]$

the coefficient of x^n is: $\binom{m}{0}\binom{m}{n} + \binom{m}{1}\binom{m}{n-1} + \binom{m}{2}\binom{m}{n-2} + \dots + \binom{m}{n}\binom{m}{0}$

since $\binom{m}{x} = \binom{m}{n-x}$ then the coefficient of x^n is

$\binom{m}{0}^2 + \binom{m}{1}^2 + \binom{m}{2}^2 + \dots + \binom{m}{n}^2$

Equating the co-efficients of x^n gives

$\binom{m}{0}^2 + \binom{m}{1}^2 + \binom{m}{2}^2 + \dots + \binom{m}{n}^2 = \binom{2m}{n}$

QUESTION 6

(a) (i) $(3+2x)^{20} = \sum_{r=0}^{20} \binom{20}{r} 3^{20-r} (2x)^r$
 so, $a_r = \binom{20}{r} 3^{20-r} 2^r$

(ii) $\frac{a_{r+1}}{a_r} = \frac{\binom{20}{r+1} 3^{19-r} 2^{r+1}}{\binom{20}{r} 3^{20-r} 2^r}$
 $= \frac{20-r}{r+1} \times \frac{2}{3}$
 $= \frac{40-2r}{3r+3}$

(iii) let $\frac{a_{r+1}}{a_r} > 1$

then, $\frac{40-2r}{3r+3} > 1$

$40-2r > 3r+3$

$5r < 37$

$r < 7\frac{2}{3}$

when $r=7$: $a_8 > a_7$

$r=6$: $a_7 > a_6$

$r=0$: $a_1 > a_0$

i.e. $a_8 > a_7 > a_6 > \dots > a_0$

if $\frac{a_{r+1}}{a_r} < 1$ then $a_8 > a_9 > \dots > a_{20}$

So the greatest coefficient is $a_8 = \binom{20}{8} 3^{12} 2^8$

$$(b)(i) \quad M_{PQ} = \frac{ap^2 - aq^2}{2ap - 2aq}$$

$$= \frac{p+q}{2}$$

$$\text{equation of PQ: } y - ap^2 = \frac{p+q}{2} (x - 2ap)$$

$$\text{so, } y = \frac{p+q}{2} x - apq$$

$$(ii) \text{ If SEPQ then when } x=0, y=a$$

$$\text{i.e. } a = D - apq$$

$$\text{so, } pq = -1$$

$$(iii) \text{ M is } (a(p+q), \frac{ap^2 + aq^2}{2})$$

$$\text{N is } (a(p+q), -a)$$

$$\text{so T is } (a(p+q), \frac{ap^2 + aq^2 - 2a}{4})$$

The locus of T is

$$x = a(p+q) \quad \text{--- (1)}$$

$$y = \frac{a}{4} (p^2 + q^2 - 2) \quad \text{--- (2)}$$

$$\text{from (i) } pq = -1, \quad y = \frac{a}{4} (p^2 + q^2 + 2pq)$$

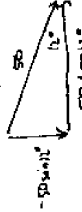
$$\text{i.e. } y = \frac{a}{4} (p+q)^2$$

$$\text{so, } y = \frac{a}{4} \frac{x^2}{a^2} \quad \text{from (1)}$$

$$\text{i.e. } x^2 = 4ay$$

QUESTION 7

$$(a)(i) \quad 180 \text{ km/h} = 50 \text{ m/s}$$



$$\dot{x} = 50 \cos 12^\circ$$

$$x = 50t \cos 12^\circ + c_1$$

$$\text{when } t=0, x=0$$

$$\text{so, } x = 50t \cos 12^\circ$$

$$\dot{y} = -10$$

$$y = -10t + c_2$$

$$\text{when } t=0, y = -50 \sin 12^\circ$$

$$\text{so, } y = -10t - 50 \sin 12^\circ$$

$$y = -5t^2 - 50t \sin 12^\circ + c_3$$

$$\text{when } t=0, y = 2.5$$

$$\text{so, } y = -5t^2 - 50t \sin 12^\circ + 2.5$$

$$(ii) \text{ when } x=6, \quad t = \frac{6}{50 \cos 12^\circ}$$

$$\text{when } t = \frac{6}{50 \cos 12^\circ}, \quad y = 1.149$$

so the ball clears the net by 15cm.

$$(iii) \text{ when } y=0, \quad 5t^2 + 50t \sin 12^\circ - 2.5 = 0$$

$$t = \frac{-50 \sin 12^\circ \pm \sqrt{(50 \sin 12^\circ)^2 + 50}}{10}$$

$$\text{when } t = \frac{-50 \sin 12^\circ + \sqrt{(50 \sin 12^\circ)^2 + 50}}{10}$$

$$x = 10.6468 \dots$$

so it lands 7.35 metres from the base line

$$(b) (i) 4 \text{ rev/min} = 8\pi \text{ rad/min}$$

$$\text{so, } \frac{d\theta}{dt} = 8\pi$$

$$(ii) \tan \theta = \frac{x}{3}$$

$$x = 3 \tan \theta$$

$$\frac{dx}{d\theta} = 3 \sec^2 \theta$$

$$\frac{dx}{dt} = \frac{dx}{d\theta} \cdot \frac{d\theta}{dt}$$

$$= 3 \sec^2 \theta \cdot 8\pi$$

$$= 24\pi \sec^2 \theta$$

$$\text{at } P \quad \theta = 0$$

$$\text{so } \frac{dx}{dt} = 24\pi \text{ km/min.}$$

$$(iii) \text{ when } x=2, \cos \theta = \frac{3}{\sqrt{13}}$$

$$\text{so, } \frac{dx}{dt} = \frac{24\pi}{\left(\frac{3}{\sqrt{13}}\right)^2}$$

$$= \frac{104\pi}{3} \text{ km/min.}$$