CSSA MATHEMATICS 3 UNIT SOLUTIONS 1999 Jimmy

QUESTION 1

(a) $2!$ (shortest and ~~largest~~ tallest) $\times 4!$ (others)

$= 2 \times 24 = \underline{\underline{48 \text{ ways}}}$.

(b) $x - 2y + 3 = 0$, $2y = x + 3$, $y = \frac{x}{2} + \frac{3}{2}$ gradient $\frac{1}{2}$

$y = mx$ gradient m

(i) $\left| \dfrac{m - (\frac{1}{2})}{1 + m(\frac{1}{2})} \right| = \tan 45°$ $\therefore \left| \dfrac{(2m-1)/2}{(2+m)/2} \right| = 1$

$\therefore \left| \dfrac{2m-1}{m+2} \right| = 1$.

(ii) $\dfrac{2m-1}{m+2} = -1$, $2m - 1 = -m - 2$, $3m = -1$, $\underline{\underline{m = -\frac{1}{3}}}$

or $\dfrac{2m-1}{m+2} = 1$, $2m - 1 = m + 2$, $\underline{\underline{m = 3}}$ ✓

(c) $\ln(x^3 + 19) = 3 \ln(x+1)$

$\therefore \ln(x^3 + 19) = \ln(x+1)^3$

$\therefore \ln(x^3 + 19) = \ln(x^3 + 3x^2 + 3x + 1)$

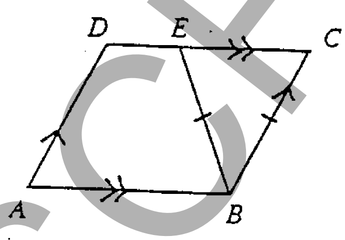
$\therefore x^3 + 19 = x^3 + 3x^2 + 3x + 1$ ✓

$\therefore 3x^2 + 3x - 18 = 0$ $\therefore x^2 + x - 6 = 0$

$\therefore (x+3)(x-2) = 0$ ✓ $\therefore$ ~~$x = -3$~~ or $\underline{\underline{x = 2}}$ ✓

($x \neq -3$ since $\ln((-3)^3 + 19)$, $\ln((-3)+1)$ are not defined)

(d) (i)



(ii) $\angle BEC = \angle BCE$ (in $\triangle BEC$ equal angles are opposite equal sides BC and E ✓

$\angle BCE = \angle BAD$ (opposite angles are equal in parallelogram ABCD) ✓

$\therefore \angle BEC = \angle BAD$ ✓

$\therefore$ ABED is a cyclic quadrilateral (exterior angle $\angle BEC$ is equal to interior opposite angle $\angle BAD$ ✓

QUESTION 2

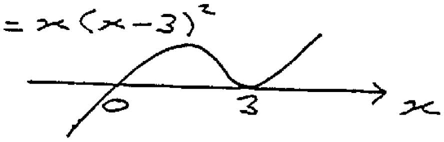
(a) $\lim\limits_{x\to 0} \dfrac{\sin 2x}{x} = 2.\lim\limits_{x\to 0}\dfrac{\sin 2x}{2x} = 2\times 1 = \underline{\underline{2}}.$ ✓

(b) $\dfrac{x^2+9}{x} \le 6 \quad \therefore \dfrac{x^2+9}{\cancel{x}} \times x^{\cancel{2}} \le 6\times x^2$ ✓

$\therefore x^3+9x \le 6x^2 \quad \therefore x^3-6x^2+9x \le 0$

$\therefore x(x^2-6x+9)\le 0 \quad \therefore x(x-3)^2 \le 0$

$y = x(x-3)^2$



$\therefore \underline{\underline{x<0 \text{ or } x=3}}$ ✓

($x \ne 0$ since $\{(0)^2+9\}/(0)$ is not defined)

(c) (i) $3x^3+3x^2-x-1 = (x+1)(3x^2-1)$ ✓

(ii) $3\tan^3\theta + 3\tan^2\theta - \tan\theta - 1 = 0 \quad (0\le\theta\le\pi)$

$\therefore (\tan\theta+1)(3\tan^2\theta-1) = 0$

$\therefore \tan\theta+1=0,\ \tan\theta=-1,\ \underline{\underline{\theta = 3\pi/4}}$ ✓

or $3\tan^2\theta - 1 = 0,\ 3\tan^2\theta = 1,\ \tan^2\theta = 1/3,$

$\tan\theta = -1/\sqrt3,\ \underline{\underline{\theta = 5\pi/6}}$ or $\tan\theta = 1/\sqrt3,\ \underline{\underline{\theta = \pi/6}}.$ ✓

(d) $x^2 = 4y.\ F(0,1),\ P(2t,t^2)$

(i) $M(x,y)$ divides FP externally in the ratio $3:1$.

$\therefore x = \dfrac{3(2t)-1(0)}{3-1} = 3t$ ✓, $y = \dfrac{3(t^2)-1(1)}{3-1} = \dfrac{3t^2-1}{2}$ ✓

$x = 3t \quad \therefore t = x/3$

$y = (3t^2-1)/2 \quad \therefore 2y = 3t^2-1$

$\therefore 2y = 3(x/3)^2 - 1 \quad \therefore 2y = \cancel{3}(x^2/\cancel{9}_3) - 1$ ✓

$\therefore 6y = x^2 - 3 \quad \therefore x^2 = 6y+3.$

$\therefore M$ moves on the parabola $x^2 = 6y+3.$

(ii) $x^2 = 6y+3 \quad \therefore x^2 = 6(y+1/2)$

$\therefore x^2 = 4(3/2)(y+1/2) \quad \therefore (x-0)^2 = 4(3/2)(y-(-1/2))$

$\therefore$ vertex $V(0,-1/2)$, focal length $a = 3/2$

$\therefore$ focus $(0, -1/2+3/2)$ ie. $\underline{\underline{(0,1)}}$ ✓

$\therefore$ directrix $y = -1/2 - 3/2$ ie. $\underline{\underline{y=-2}}$

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QUESTION 3

(a) $y = \tan^{-1} 1/x$ $\therefore \frac{dy}{dx} = \frac{1}{1+(1/x)^2} \cdot (-1/x^2)$

$\therefore$ when $x = 1$, $dy/dx = 1/(1+1) \cdot (-1) = -1/2$

$\therefore$ the gradient of the tangent is $\underline{\underline{-1/2}}$

(b) $f(x) = \frac{x+1}{x+2}$ $\therefore y = \frac{x+1}{x+2}$

interchange x and y $\therefore x = \frac{y+1}{y+2}$

$\therefore x(y+2) = y+1$ $\therefore xy + 2x = y+1$

$\therefore xy - y = 1 - 2x$ $\therefore y(x-1) = 1-2x$

$\therefore y = \frac{1-2x}{x-1}$ $\therefore \underline{\underline{f^{-1}(x) = \frac{1-2x}{x-1}}}$

(c) $\frac{dy}{dx} = 2\cos^2 x + 1$ $\therefore y = \int (2\cos^2 x + 1)\,dx$

$\therefore y = \int (\cos 2x + 1 + 1)\,dx$ $\therefore y = \int (2 + \cos 2x)\,d$

$\therefore y = 2x + 1/2 \sin 2x + c$

when $x = \pi$, $y = \pi$ $\therefore \pi = 2\pi + 1/2 \sin 2\pi + c$

$\therefore \pi = 2\pi + 0 + c$ $\therefore c = -\pi$

$\therefore y = 2x + 1/2 \sin 2x - \pi$

$\therefore$ when $x = 2\pi$, $y = 4\pi + 1/2 \sin 4\pi - \pi$

$\therefore y = 4\pi + 0 - \pi$ $\therefore \underline{\underline{y = 3\pi}}$

(d) $\int_1^{100} \frac{1}{x + 2\sqrt{x}}\,dx$

$x = u^2 \ (u > 0)$

$\therefore dx/du = 2u$, $dx = 2u\,du$

when $x = 1$, $u = 1$

when $x = 100$, $u = 10$

$= \int_1^{10} \frac{1}{u^2 + 2u} \cdot 2u\,du$

$= \int_1^{10} \frac{1}{\cancel{u}(u+2)} \cdot 2\cancel{u}\,du = 2\int_1^{10} \frac{1}{u+2}\,du$

$= 2\,[\ln(u+2)]_1^{10} = 2(\ln 12 - \ln 3)$

$= 2 \ln 12/3 = 2 \ln 4$

$= \ln 4^2 = \underline{\underline{\ln 16}}$

QUESTION 4

(a) $\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx = \int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{2^2-x^2}}\,dx$

$= \left[\sin^{-1} x/2\right]_{\sqrt{2}}^{\sqrt{3}}$ ✓ $= \sin^{-1}\sqrt{3}/2 - \sin^{-1}\sqrt{2}/2$

$= \pi/3 - \pi/4 = \underline{\pi/12}$ ✓

(b)(i) $t = x^2 - 3x + 2 \quad \therefore dt/dx = 2x - 3$

$v = dx/dt = 1/dt/dx \quad \therefore \underline{v = 1/(2x-3)}$ ✓

(ii) $a = d/dx\,(\tfrac{1}{2}v^2) = d/dx\,(\tfrac{1}{2}\cdot 1/(2x-3)^2)$

$\therefore a = \tfrac{1}{2}\cdot -2/(2x-3)^3 \cdot 2 \quad \therefore \underline{a = -2/(2x-3)^3}$ ✓

(c) $y = 2\cos^{-1}(1-x)$

(i) domain : $-1 \le 1-x \le 1$

$\therefore -2 \le -x \le 0 \quad \therefore \underline{0 \le x \le 2}$

range : $0 \le \cos^{-1}(1-x) \le \pi$

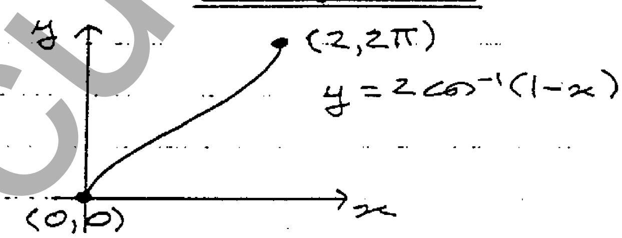
$\therefore 0 \le 2\cos^{-1}(1-x) \le 2\pi \quad \therefore \underline{0 \le y \le 2\pi}$

(ii) when $x = 0$

$y = 2\cos^{-1}(1) = 0$

when $x = 2$

$y = 2\cos^{-1}(-1) = 2\pi$



(d) $r = \frac{1+3t}{1+t} \quad \therefore \frac{dr}{dt} = \frac{(1+t)(3)-(1+3t)(1)}{(1+t)^2} = \frac{2}{(1+t)^2}$ ✓

When $r = 2$, $\frac{1+3t}{1+t} = 2$

$\therefore 1+3t = 2+2t \quad \therefore t = 1.$ ✓

$A = \pi r^2 \quad \therefore \frac{dA}{dt} = \frac{dA}{dr}\cdot\frac{dr}{dt} = 2\pi r \cdot \frac{2}{(1+t)^2}$

When $r = 2$, $t = 1$

$\frac{dA}{dt} = 2\pi(2)\cdot\frac{2}{(1+(1))^2} = 2\pi$

$\therefore$ the area of the oil spill is increasing at a rate of $\underline{2\pi \text{ kilometres}^2/\text{hour}}$. ✓

7(c) RTP $1\times 2^0 + 2\times 2 + 3\times 2^2 + \dots + n\times 2^{n-1} = 1 + (n-1).2^n$

<u>Step 1.</u> <u>$n=1$</u>

$LHS = 1\times 2^0 = 1$

$RHS = 1 + (1-1).2^1 = 1$

$\therefore LHS = RHS$

<u>Step 2.</u> Assume hypothesis is true for <u>$n=k$</u> $(k>1)$

$\therefore 1\times 2^0 + 2\times 2 + 3\times 2^2 + \dots + k\times 2^{k-1} = 1 + (k-1).2^k$

For <u>$n=k+1$</u>

$LHS = 1\times 2^0 + 2\times 2 + 3\times 2^2 + \dots + k\times 2^{k-1} + (k+1)\times 2^k$

$= 1 + (k-1).2^k + (k+1).2^k$ (using hypothesis)

$= 1 + k.2^k - 2^k + k.2^k + 2^k$

$= 1 + 2.k.2^k$

$= 1 + k.2^{k+1}$

$RHS = 1 + (k+1-1).2^{k+1}$

$= 1 + k.2^{k+1}$

$\therefore LHS = RHS.$

<u>Step 3</u> Hence if the hypothesis is true for $n=k$ then it is true for $n=k+1$. Since it is true for $n=1$ it is true for $n=1+1=2$ and so on for all integers $n\geq 1$.

QUESTION 5

a) $f(x) = \ln x / x \quad (x > 0)$

(i) $f'(x) = \{(x)(1/x) - (\ln x)(1)\}/x^2 = (1-\ln x)/x^2$ ✓

When $f'(x) = 0$, $(1-\ln x)/x^2 = 0$, $1-\ln x = 0$, $\ln x = 1$, $x = e$

When $x = e$, $y = \ln e / e = 1/e$

When $0 < x < e$, $\ln x < 1$ and $f'(x) = (1-\ln x)/x^2 > 0$

When $x > e$, $\ln x > 1$ and $f'(x) = (1-\ln x)/x^2 < 0$

$\therefore$ $(e, 1/e)$ is a maximum turning point. ✓

(ii) $f(e)$ is a local maximum and $f'(x) < 0$ for $x > e$ so that $f(x)$ is a decreasing function for $x > e$ ✓ $\therefore f(\pi) < f(e)$

$\therefore \ln \pi / \pi < \ln e / e$ ✓ $\therefore e \ln \pi < \pi \ln e$

$\therefore \ln \pi^e < \ln e^\pi \quad \therefore \pi^e < e^\pi$.

(iii) $\ln x / x = -2 \quad \therefore \ln x = -2x \quad \therefore \ln x + 2x = 0$ ✓

$P(x) = \ln x + 2x$, $P'(x) = 1/x + 2$

$\therefore$ with initial approximation of $X = 0.5$, improved approximation

$= 0.5 - \dfrac{P(0.5)}{P'(0.5)} = 0.5 - \dfrac{\ln 0.5 + 2(0.5)}{1/(0.5) + 2} = \underline{0.42}$ (to 2 d.p.) ✓

b) (i) $dM/dt < 0$, $dM/dt \propto (M - 1000)$ ✓

$\therefore dM/dt = k(M-1000)$ $(k<0)$ or $dM/dt = -k(M-1000)$ $(k>0)$

If $M = 1000 + Ae^{-kt}$

$dM/dt = 0 + A(-ke^{-kt}) = -k(Ae^{-kt}) = -k(M-1000)$ ✓

$\therefore M = 1000 + Ae^{-kt}$ is a solution of $dM/dt = -k(M-1000)$.

(ii) When $t = 0$, $M = 49000$. $\therefore 49000 = 1000 + Ae^{-k(0)}$

$\therefore 49000 = 1000 + A \quad \therefore \underline{A = 48000}$ ✓

When $t = 2$, $M = 25000 \quad \therefore 25000 = 1000 + 48000e^{-k(2)}$

$\therefore 24000 = 48000e^{-2k} \quad \therefore e^{2k} = 2 \quad \therefore \underline{k = \tfrac{1}{2}\ln 2}$ ✓

(iii) $dM/dt = -k(M-1000)$; when $t = 0$, $M = 49000$

$\therefore$ initial rate of losing value $= -k(49000-1000) = -48000k$

When $dM/dt = 1/4(-48000k) = -12000k$,

$-k(M-1000) = -12000k$, $M - 1000 = 12000$, $\underline{M = \$13000}$ ✓

When $M = 13000$, $13000 = 1000 + 48000e^{-kt}$

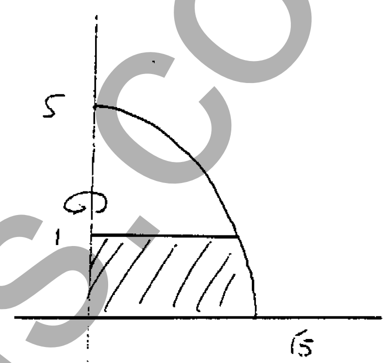
$\therefore 12000 = 48000e^{-kt} \quad \therefore e^{kt} = 4 \quad \therefore kt = \ln 4$

$\therefore t = \dfrac{\ln 4}{k} = \dfrac{\ln 4}{\frac{1}{2}\ln 2} = \underline{4 \text{ years}}$

6 (a) (i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ ✓

(ii) $\sin^{-1}(-x) = -\sin^{-1}(x)$ ✓

(b) $V = \pi \int_a^b [f(y)]^2 dy$



$y = 5 - x^2$

$\therefore x^2 = 5 - y$ ✓

$\therefore V = \pi \int_0^1 (5-y)\,.dy$ ✓

$= \pi \left[5y - \frac{y^2}{2} \right]_0^1$ ✓

$= \pi \left\{ \left(5 - \frac{1}{2}\right) - 0 \right\}$

$= \frac{9\pi}{2}$ cubic units. ✓

QUESTION 6

(a) In the expansion of $(1+x)^{14}$ the coefficients of x^4, x^5, x^6 are ${}^{14}C_4 = 1001$, ${}^{14}C_5 = 2002$, ${}^{14}C_6 = 3003$ which are consecutive terms in an arithmetic sequence with common difference 1001.

(b) In any one throw $P(\text{heads}) = p$.

$P(3 \text{ heads in } 6 \text{ throws}) = 2P(2 \text{ heads in } 6 \text{ throws})$

$\therefore {}^6C_3\, p^3(1-p)^3 = 2\,{}^6C_2\, p^2(1-p)^4$

$\therefore 20p^3(1-p)^3 = 2.15p^2(1-p)^4$

$\therefore 20p^3(1-p)^3 = 30p^2(1-p)^4 \quad \therefore 2p = 3(1-p)$

$\therefore 2p = 3 - 3p \quad \therefore 5p = 3 \quad \therefore \underline{\underline{p = 3/5}}$

(c) $x = 2\sin 3t - 2\sqrt{3}\cos 3t$

(i) $2\sin 3t - 2\sqrt{3}\cos 3t \equiv R\sin(3t-\alpha)$

$\boxed{= 4\left(\tfrac{1}{2}\sin 3t - \tfrac{\sqrt{3}}{2}\cos 3t\right)}$ ✓ $\quad \equiv R(\sin 3t\cos\alpha - \cos 3t\sin\alpha)$

$\equiv (R\cos\alpha)\sin 3t - (R\sin\alpha)\cos 3t$

$\therefore R\cos\alpha = 2 \quad (1) \qquad R\sin\alpha = 2\sqrt{3} \quad (2)$

$(1)^2 \quad R^2\cos^2\alpha = 4 \qquad R^2(\cos^2\alpha + \sin^2\alpha) = 16$

$(2)^2 \quad R^2\sin^2\alpha = 12 \quad + \qquad R^2 = 16, \quad R = 4$

$\dfrac{(2)}{(1)} \quad \dfrac{R\sin\alpha}{R\cos\alpha} = \dfrac{2\sqrt{3}}{2} \qquad \tan\alpha = \sqrt{3}, \quad \alpha = \pi/3$ ✓

$\therefore \underline{\underline{x = 4\sin(3t - \pi/3)}}$ ✓

(ii) $x = 4\sin(3t - \pi/3)$; when $t = 0$, $x = -2\sqrt{3}$

$v = \dot{x} = 12\cos(3t - \pi/3)$; when $t = 0$, $v = 6$

$a = \ddot{x} = -36\sin(3t - \pi/3)$; when $t = 0$, $a = 18\sqrt{3}$

$\therefore$ initially the particle is $2\sqrt{3}$ metres to the left of O moving to the right at a speed of $6\ ms^{-1}$ and speeding up at a rate of $18\sqrt{3}\ ms^{-2}$.

(iii) When $x = -2$, $4\sin(3t - \pi/3) = -2$

$\therefore \sin(3t - \pi/3) = -1/2 \qquad \therefore 3t - \pi/3 = -\pi/6, \ldots$

$\therefore 3t = -\pi/6 + \pi/3, \ldots \qquad \therefore 3t = \pi/6, \ldots$

$\therefore t = \pi/18, \ldots \qquad$ When $t = \pi/18$, $v = 6\sqrt{3} > 0$

$\therefore$ the first time is after $\underline{\underline{\pi/18 \text{ seconds}}}$

7 (a) (i) $f^2 = x^2 + 9$

$\therefore f = \sqrt{x^2+9}$ ✓

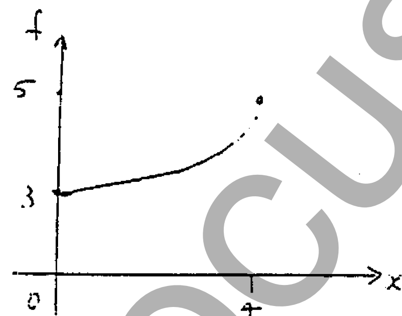
(ii) $D(f): 0 \leq x \leq 4$ ✓

(iii) $R(f): 3 \leq f \leq 5$ ✓

(iv) Greatest $f = 5$ ✓

Least $f = 3$ ✓

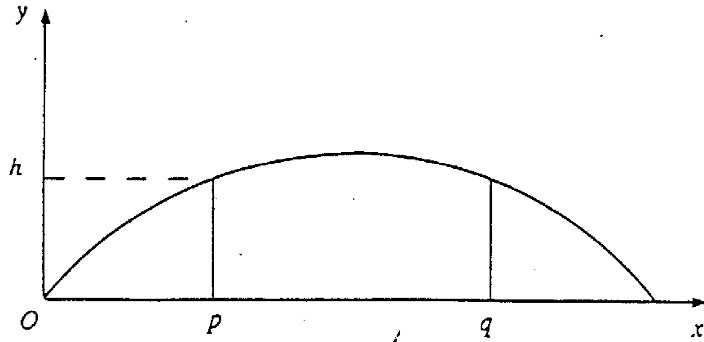
(v)



✓ endpoints match ii, iii (or iv)

QUESTION 7

(b)



(i) $x = (V\cos\alpha)t$ (1), $y = (V\sin\alpha)t - \frac{1}{2}gt^2$

$y = (V\sin\alpha)t - 5t^2$ ✓ (2)

(ii) When $x = p$, $y = h$

$\therefore$ in (1) $p = (V\cos\alpha)t$ $\therefore t = p/V\cos\alpha$ ✓

$\therefore$ in (2) $h = (V\sin\alpha)t - 5t^2$

$\therefore h = (V\sin\alpha)(p/V\cos\alpha) - 5(p/V\cos\alpha)^2$

$\therefore h = p\tan\alpha - 5p^2/(V^2\cos^2\alpha)$

$\therefore h = p\tan\alpha - (5p^2/V^2)\sec^2\alpha$

$\therefore h = p\tan\alpha - 5p^2/V^2\,(1+\tan^2\alpha)$

$\therefore 5p^2/V^2\,(1+\tan^2\alpha) = p\tan\alpha - h$ ✓

$$\therefore V^2 = \frac{5p^2(1+\tan^2\alpha)}{p\tan\alpha - h}$$

(iii) When $x = q$, $y = h$

$$\therefore \text{similarly } V^2 = \frac{5q^2(1+\tan^2\alpha)}{q\tan\alpha - h}$$

$$\therefore \frac{5p^2\cancel{(1+\tan^2\alpha)}}{p\tan\alpha - h} = \frac{5q^2\cancel{(1+\tan^2\alpha)}}{q\tan\alpha - h}$$ ✓

$\therefore p^2(q\tan\alpha - h) = q^2(p\tan\alpha - h)$

$\therefore p^2q\tan\alpha - p^2h = pq^2\tan\alpha - q^2h$

$\therefore p^2h - q^2h = p^2q\tan\alpha - pq^2\tan\alpha$

$\therefore (p^2 - q^2)h = (p^2q - pq^2)\tan\alpha$

$\therefore (p+q)\cancel{(p-q)}h = pq\cancel{(p-q)}\tan\alpha$

$$\therefore \tan\alpha = \frac{h(p+q)}{pq}$$