CATHOLIC SECONDARY SCHOOLS' ASSOCIATION OF NEW SOUTH WALES

YEAR TWELVE FINAL TESTS 1999

MATHEMATICS

3/4 UNIT

(i.e. 3 UNIT COURSE – ADDITIONAL PAPER:
4 UNIT COURSE – FIRST PAPER)

Afternoon session

Friday 13 August 1999

Time allowed – two hours

EXAMINERS
Graham Arnold, Terra Sancta College, Nirimba
Sandra Hayes, Aquinas College, Menai
Frank Reid, St Ursula's College, Kingsgrove

DIRECTIONS TO CANDIDATES:

- ALL questions may be attempted.
- ALL questions are of equal value.
- All necessary working should be shown in every question.
- Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.
- Standard integrals are printed at the end of the exam paper.

Students are advised that this is a Trial Examination only and cannot in any way guarantee the content or the format of the Higher School Certificate Examination. However, the committees responsible for the preparation of these 'Trial Examinations' do hope that they will provide a positive contribution to your preparation for the final examinations.

Marks

Question 1 **Begin a new page**

(a) If $f(x) = x^3 + 3x^2 - 10x - 24$ calculate $f(-2)$. 1

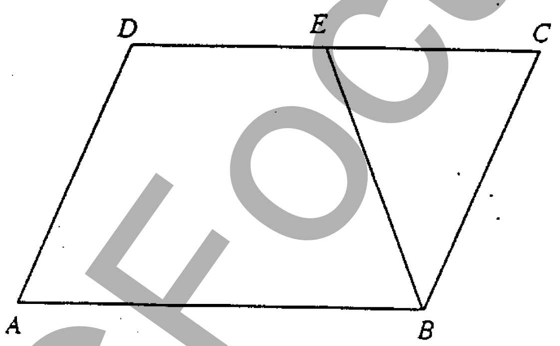
(b) The acute angle between the line $x - 2y + 3 = 0$ and the line $y = mx$ is 45°. 3

(i) Show that $\left|\frac{2m-1}{m+2}\right| = 1$

(ii) Find the possible values of m.

(c) Solve the equation $\ln(x^3 + 19) = 3\ln(x+1)$. 3

(d) 5



$ABCD$ is a parallelogram. E is the point on CD such that $BE = BC$.

(i) Copy the diagram showing the above information.

(ii) Show that $ABED$ is a cyclic quadrilateral.

Marks

Question 2 Begin a new page

(a) Find $\lim_{x \to 0} \frac{\sin 2x}{x}$ 1

(b) Solve the inequality $\frac{x^2+9}{x} \leq 6$ 3

(c) (i) Factorise $3x^3+3x^2-x-1$ 3

(ii) Solve the equation $3\tan^3\theta+3\tan^2\theta-\tan\theta-1=0$ for $0 \leq \theta \leq \pi$

(d) $P(2t, t^2)$ is a point on the parabola $x^2=4y$ with focus F. The point M divides the interval FP externally in the ratio 3 : 1. 5

(i) Show that as P moves on the parabola $x^2=4y$, then M moves on the parabola $x^2=6y+3$.

(ii) Find the coordinates of the focus and the equation of the directrix of the locus of M.

Question 3 Begin a new page

(a) Find the gradient of the tangent to the curve $y=\tan^{-1}\frac{1}{x}$ at the point on the curve where $x=1$. 2

(b) A function is given by the rule $f(x)=\frac{x+1}{x+2}$. Find the rule for the inverse function $f^{-1}(x)$. 2

(c) At any point on the curve $y=f(x)$ the gradient function is given by $\frac{dy}{dx} = 2\cos^2 x+1$. If $y=\pi$ when $x=\pi$, find the value of y when $x=2\pi$. 4

(d) Use the substitution $x=u^2$, $u>0$, to express the value of $\int_1^{100} \frac{1}{x+2\sqrt{x}}\,dx$ in the form $\ln a$ for some constant $a>0$. 4

Marks

Question 4 Begin a new page

(a) Find the exact value of $\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\, dx$. 2

(b) A particle is moving in a straight line. At time t seconds its displacement x metres from a fixed point O on the line is such that $t = x^2 - 3x + 2$. 2

(i) Find an expression for its velocity v in terms of x.

(ii) Find an expression for its acceleration a in terms of x.

(c) Prove by mathematical induction that 4
$1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + \ldots + n \times 2^{n-1} = 1 + (n-1)2^n$
for all integers $n \geq 1$.

(d) The radius r kilometres of a circular oil spill at time t hours after it was first observed is given by $r = \frac{1+3t}{1+t}$. Find the exact rate of increase of the area of the oil spill when the radius is 2 kilometres. 4

Question 5 Begin a new page

(a) Consider the function $f(x) = \frac{\ln x}{x}$. 6

(i) Find the coordinates and the nature of the stationary point on the curve $y = f(x)$.

(ii) Explain why $f(\pi) < f(e)$ and hence show that $\pi^e < e^\pi$.

(iii) $P(X, -2)$ is a point on the curve $y = f(x)$. Starting with an initial approximation of $X = 0{\cdot}5$, use one application of Newton's method to find an improved approximation to the value of X, giving the answer correct to 2 decimal places.

Marks

Question 5 (Cont)

(b) A machine which initially costs \$49 000 loses value at a rate proportional to the difference between its current value \$$M$ and its final scrap value \$1000. After 2 years the value of the machine is \$25 000. 6

(i) Explain why $\frac{dM}{dt} = -k(M - 1000)$ for some constant $k > 0$, and verify that $M = 1000 + Ae^{-kt}$, A constant, is a solution of this equation.

(ii) Find the exact values of A and k.

(iii) Find the value of the machine, and the time that has elapsed, when the machine is losing value at a rate equal to one quarter of the initial rate at which it loses value.

Question 6 Begin a new page

(a) (i) Find the value of $\sin^{-1}x + \cos^{-1}x$. 2

(ii) Explain why the function $y = \sin^{-1}x$ is odd.

(b) Find the volume of the solid formed when the area bounded by the curve $y = 5 - x^2$ for $x \geq 0$, the y axis and the line $y = 1$ is rotated about the y axis. 4

(c) A particle moving in a straight line is performing Simple Harmonic Motion. At time t seconds its displacement x metres from a fixed point O on the line is given by $x = 2\sin 3t - 2\sqrt{3}\cos 3t$. 6

(i) Express x in the form $x = R\sin(3t - \alpha)$ for some constants $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

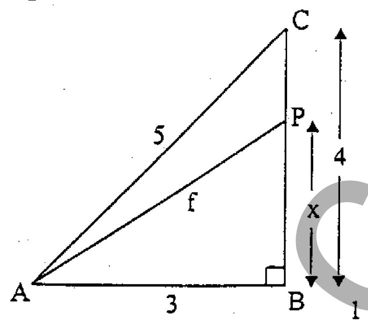
(ii) Describe the initial motion of the particle in terms of its initial position, velocity and acceleration.

(iii) Find the exact value of the first time that the particle is 2 metres to the left of O and moving towards O.

Marks

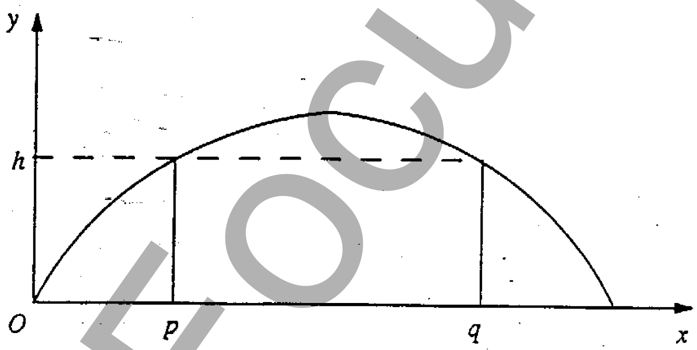
Question 7 Begin a new page

(a) ABC is a right angled triangle at B. P is a point on BC. The distance AP is f units. The point P moves on the line segment BC. **6**



(i) Write f as a function of x.
(ii) Find the domain D(f) of f(x).
(iii) Find the range of f(x).
(iv) What are the greatest and the least values of f(x)?
(v) Sketch the function of f(x).

(b) **6**



A particle is projected with velocity V ms^{-1} from a point O at an angle of elevation α. Axes Ox and Oy are taken horizontally and vertically through O. The particle just clears two vertical chimneys of height h metres at horizontal distances of p metres and q metres from O. The acceleration due to gravity is taken as 10 ms^{-2} and air resistance is ignored.

(i) Write down expressions for the horizontal displacement x and the vertical displacement y of the particle after time t seconds.

(ii) Show that $V^2 = \dfrac{5p^2\left(1+\tan^2\alpha\right)}{p\tan\alpha - h}$.

(iii) Show that $\tan\alpha = \dfrac{h(p+q)}{pq}$.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$