S

SYDNEY BOYS HIGH SCHOOL

MOORE PARK, SURRY HILLS

1998

HIGHER SCHOOL CERTIFICATE

ASSESSMENT TASK #3

Mathematics Extension 1

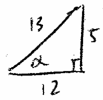
Sample Solutions

1998 Ext 1 Task # 3

(1) (a) (i) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$.

(ii) $\lim_{x\to 0} \frac{\tan 3x}{4x} = \frac{3}{4} \lim_{x\to 0} \frac{\tan 3x}{3x} = \frac{3}{4}$

(iii) let $\alpha = \tan^{-1}\frac{5}{12} \Rightarrow \tan\alpha = \frac{5}{12}$



$\therefore \cos\left(\tan^{-1}\frac{5}{12}\right) = \cos\alpha = \frac{12}{13}$

(b) $x = \cos 4t - \sin 4t$

$\dot{x} = -4\sin 4t - 4\cos 4t$

$\ddot{x} = -16\cos 4t + 16\sin 4t$

$= -16(\cos 4t - \sin 4t)$

$= -16x$

(c) $\int \frac{dx}{1+3x}$

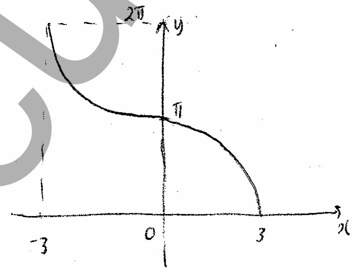
$= \frac{1}{3}\int \frac{3\,dx}{1+3x}$

$= \frac{1}{3}\ln|1+3x| + c$

(d) $f(x) = 2\cos^{-1}\frac{x}{3}$

D: $-1 \le \frac{x}{3} \le 1 \Rightarrow -3 \le x \le 3$

R: $0 \le \frac{y}{2} \le \pi \Rightarrow 0 \le y \le 2\pi$



(2) (a) (i) $\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\left(\frac{x}{2}\right)\Big]_0^1$

$= \sin^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{6}$

(ii) $\int_0^{\pi/2} \cos^2 x \sin x\,dx$

$u = \cos x$

$du = -\sin x\,dx$

$x = 0, u = 1$

$x = \frac{\pi}{2}, u = 0$

$= -\int_1^0 u^2\,(du)$

$= \int_0^1 u^2\,du = \frac{1}{3}u^3\Big]_0^1$

$= \frac{1}{3}$

(iii) $\int_{\frac{1}{4}}^{\frac{\sqrt{3}}{4}} \frac{1}{1+16x^2}\,dx$

$= \frac{1}{16}\int_{\frac{1}{4}}^{\frac{\sqrt{3}}{4}} \frac{1}{\frac{1}{16}+x^2}\,dx$

$= \frac{1}{16} \times 4\tan^{-1}(4x)\Big]_{\frac{1}{4}}^{\frac{\sqrt{3}}{4}}$

$= \frac{1}{4}\left[\tan^{-1}\sqrt{3} - \tan^{-1}(1)\right] = \frac{1}{4}\left[\frac{\pi}{3} - \frac{\pi}{4}\right] = \frac{\pi}{48}$

2 (b) (i) $\dfrac{d\left(\tan(x^2)\right)}{dx} = 2x\sec^2(x^2)$

(ii) $\dfrac{d\left(\frac{e^{-x}}{x^2}\right)}{dx} = \dfrac{x^2(-e^{-x}) - e^{-x}(2x)}{x^4}$

$= \dfrac{-\cancel{x}e^{-x}(x+2)}{x^4}$

$= \dfrac{-e^{-x}(x+2)}{x^3}$

(c) $f(x) = e^{-x^2}$

$f'(x) = -2xe^{-x^2}$

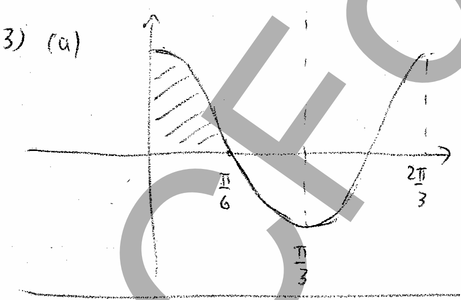
$f''(x) = e^{-x^2}(-2) + (-2x)(-2xe^{-x^2})$

$= -2e^{-x^2}(1-2x^2)$

$f''(1) = -2e^{-1}(-1)$

$= \dfrac{2}{e}$

(3) (a)



$V = \pi\int_0^{\pi/6} \cos^2 3x \, dx$

$= \dfrac{\pi}{2}\int_0^{\pi/6} 2\cos^2 3x \, dx$

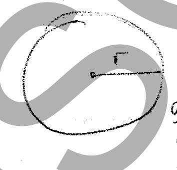
$= \dfrac{\pi}{2}\int_0^{\pi/6} (1+\cos 6x)\, dx$

$= \dfrac{\pi}{2}\left[x + \dfrac{1}{6}\sin 6x\right]_0^{\pi/6}$

$= \dfrac{\pi}{2}\left(\dfrac{\pi}{6} + \dfrac{1}{6}\sin\pi\right)$

$= \dfrac{\pi^2}{12}$ c.u.

b)



$A = \pi r^2$

$\dfrac{dA}{dt} = 4$

$\therefore \dfrac{dA}{dt} = 2\pi r \cdot \dfrac{dr}{dt}$

$(r=3)$ $4 = 2\pi(3)\dfrac{dr}{dt}$

$\dfrac{dr}{dt} = \dfrac{2}{3\pi}$ cm/s.

(3) (c) (i) RHS $= \dfrac{d(\frac{1}{2}v^2)}{dx}$

$= \dfrac{d(\frac{1}{2}v^2)}{dv} \times \dfrac{dv}{dx}$

$= v\dfrac{dv}{dx}$

$= \dfrac{dx}{dt} \cdot \dfrac{dv}{dx}$

$= \dfrac{dv}{dt}$

$= a$

$= \ddot{x}$

$=$ LHS

(ii) $\ddot{x} = -100x$

$\therefore \dfrac{d(\frac{1}{2}v^2)}{dx} = -100x$

$\therefore \frac{1}{2}v^2 = -50x^2 + C$

$(v = 10, x = 2)$

$v^2 = -100x^2 + k$

$100 = -100 \times 4 + k$

$\therefore k = 500$

$\therefore v^2 = 500 - 100x^2$

$= 100(5 - x^2)$

max speed when $\ddot{x} = 0 \Rightarrow x = 0$

$\therefore v^2 = 500$

$\therefore |v| = \sqrt{500} = 10\sqrt{5}$ units/sec.

(4) $y = x\ln x - 1$, $x > 0$

(i) $y' = x \times \frac{1}{x} + \ln x = 0$

$\therefore \ln x = -1 \Rightarrow x = e^{-1}$

$y'' = \frac{1}{x}$ at $x = e^{-1}$, $y'' > 0$ $\therefore$ min.

$x = e^{-1}$ $y = e^{-1}\ln e^{-1} - 1$

$= -e^{-1} - 1$

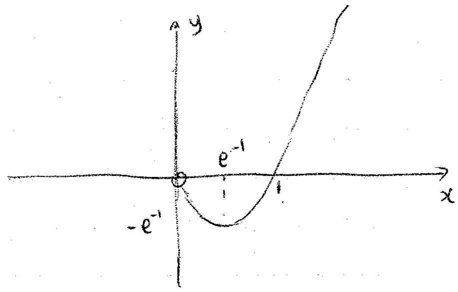
$\therefore (e^{-1}, -e^{-1} - 1)$

(ii) $x_0 = 2$ $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)} = 2 - \dfrac{2\ln 2 - 1}{1 + \ln 2} \doteqdot 1.8$

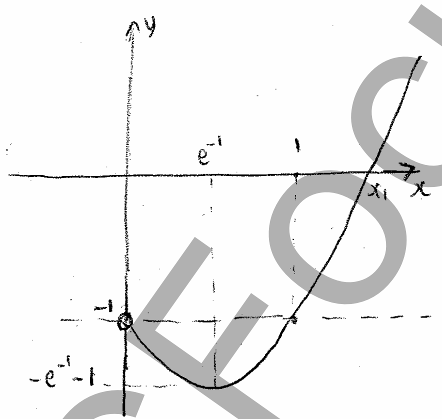
(iii) $y'' = \frac{1}{x} > 0$ for $x > 0$ $\therefore$ concave up.

4(a)(iv) sketch $y = x\ln x - 1$

I: First sketch $y = x \ln x$



II: shift curve down 1 unit



(b) $\frac{dP}{dt} = -kP$

$\therefore P = P_0 e^{-kt}$

$LHS = \frac{dP}{dt} = -k(P_0 e^{-kt})$

$= -kP$

$= RHS$

(5) (a) $a = 8e^{-4t}$

(i) $\therefore \frac{dv}{dt} = -2e^{-4t} + c$

$\therefore v = -2e^{-4t} + c$

$(t=0, \ v=2)$

$\therefore 2 = -2e^{0} + c$

$\therefore c = 4$

$\therefore v = 4 - 2e^{-4t}$

(ii) $3 = 4 - 2e^{-4t}$

$\therefore -1 = -2e^{-4t}$

$\therefore e^{-4t} = \frac{1}{2}$

$\therefore -4t = \ln(\frac{1}{2}) = -\ln 2$

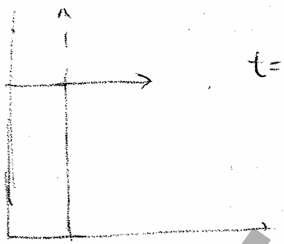
$\therefore t = \frac{1}{4}\ln 2$

(iii) as $t \to \infty$ $e^{-4t} \to 0$

$\therefore v \to 4$

(b)

(i)



$t=0, \ x=0, \ \dot{x}=u$

$y=h, \ \dot{y}=0$

$\boxed{\ddot{x} = 0}$

$\therefore \dot{x} = c$

$\therefore \boxed{\dot{x} = u}$

$\therefore x = ut + k$

$\therefore \boxed{x = ut}$ $(t=0, x=0)$

$\boxed{\ddot{y} = -g}$

$\therefore \dot{y} = -gt + c_1$

$\boxed{\dot{y} = -gt}$

$\therefore y = -\frac{g}{2}t^2 + k_1$

$\therefore \boxed{y = h - \frac{g}{2}t^2}$ $(t=0, y=h)$

(ii) $y=0 \Rightarrow h - \frac{g}{2}t^2 = 0$

$\therefore \frac{g}{2}t^2 = h$

$t^2 = \frac{2h}{g}$

$\therefore t = \sqrt{\frac{2h}{g}}$ secs.

(iii) $R = u \times \sqrt{\frac{2h}{g}}$

$= u\sqrt{\frac{2h}{g}}$ m

(6) (a) $\frac{d\theta}{dt} = \frac{30-\theta}{20} = -\frac{1}{20}(\theta - 30)$

(i) $\therefore \theta = 30 + \theta_0 e^{-\frac{1}{20}t}$ (Newton's Law of cooling)

$t = 0,\ \theta = 10 \Rightarrow \theta_0 = -20$

$\therefore \theta = 30 - 20e^{-\frac{1}{20}t}$

(ii) $t = 60$ mins, $\theta = ?$

$\therefore \theta = 30 - 20e^{-3}$

$\doteqdot 29^\circ$

(iii) $\theta = 20,\ t = ?$

$\therefore 20 = 30 - 20e^{-\frac{1}{20}t}$

$\therefore -10 = -20e^{-1/20 t}$

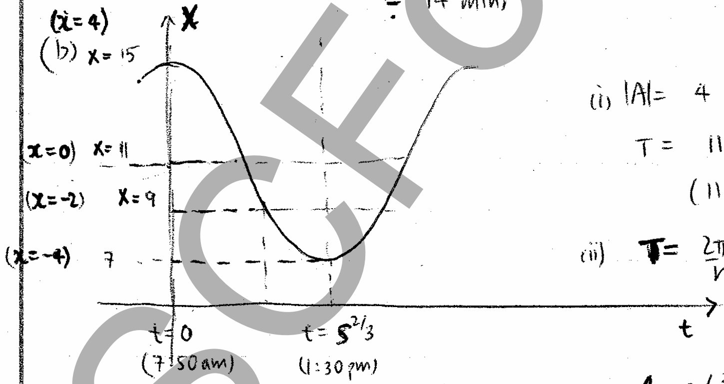
$\therefore e^{-1/20 t} = \frac{1}{2}$

$\therefore -\frac{1}{20}t = \ln\left(\frac{1}{2}\right) = -\ln 2$

$\therefore t = 20 \ln 2$ mins.

$\doteqdot 14$ mins

(b)



(i) $|A| = 4$ metres

$T = 11\frac{1}{3}$ hours $= \frac{34}{3}$ hours

(11 hrs 20 mins)

(ii) $T = \frac{2\pi}{n}$ $\therefore n = \frac{2\pi}{T} = \frac{2\pi \times 3}{34}$

$= \frac{3\pi}{17}$

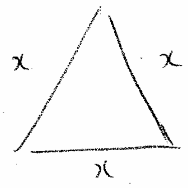
$x = 4\cos\left(\frac{3\pi}{17}t\right)$

$\therefore -2 = 4\cos\left(\frac{3\pi}{17}t\right)$

$\therefore \frac{3\pi}{17}t = \frac{2\pi}{3}, \frac{4\pi}{3}$

$t = \frac{34}{9}, \frac{68}{9}$ hours later $\therefore$ $\boxed{11:37 \text{ am to } 3:23 \text{ pm}}$

(6) (c)



$P = 3x$

$A = \frac{1}{2}x^2 \sin 60°$

$= \frac{x^2\sqrt{3}}{4}$

method 1: $x = \frac{P}{3}$

$\therefore A = \left(\frac{P}{3}\right)^2 \times \frac{\sqrt{3}}{4}$

$= \frac{\sqrt{3}P^2}{36}$

$\boxed{\frac{dA}{dP} = \frac{\sqrt{3}}{36}P}$

method 2:

$\frac{dA}{dP} = \frac{dA}{dx} \times \frac{dx}{dP}$

$= \frac{2x\sqrt{3}}{4} \times \frac{1}{3}$

$= \frac{x\sqrt{3}}{12} = \frac{3x\sqrt{3}}{36}$

$= \boxed{\frac{\sqrt{3}P}{36}}$