Question One

(a) $y = \dfrac{\tan x}{e^{2x}}$ $\begin{matrix}(u)\\(v)\end{matrix}$ $\quad y' = \dfrac{vu' - uv'}{v^2}$

$$\frac{dy}{dx} = \frac{e^{2x}.\sec^2 x - \tan x.2e^{2x}}{e^{4x}}$$

$$= \frac{\sec^2 x - 2\tan x}{e^{2x}}$$

$$= \frac{\tan^2 x - 1 - 2\tan x}{e^{2x}}$$

$$= \frac{(\tan x - 1)^2}{e^{2x}}$$

$$= \frac{(1 - \tan x)^2}{e^{2x}}$$

(b) $\sin 2x = \tan x \qquad 0 \le x \le \pi.$

$2\sin x\cos x = \dfrac{\sin x}{\cos x}$

$2\sin x\cos^2 x - \sin x = 0$

$\sin x(2\cos^2 x - 1) = 0$

$\therefore \sin x = 0$

$\therefore x = 0, \pi$

or $2\cos^2 x - 1 = 0$

$\cos x = \pm\frac{1}{\sqrt{2}}$

$\therefore x = \frac{\pi}{4}, \frac{3\pi}{4}$

(c). $y = 4\cos^{-1}\left(\frac{x}{3}\right)$.

(i)

now: domain is $-1 \le \frac{x}{3} \le 1$

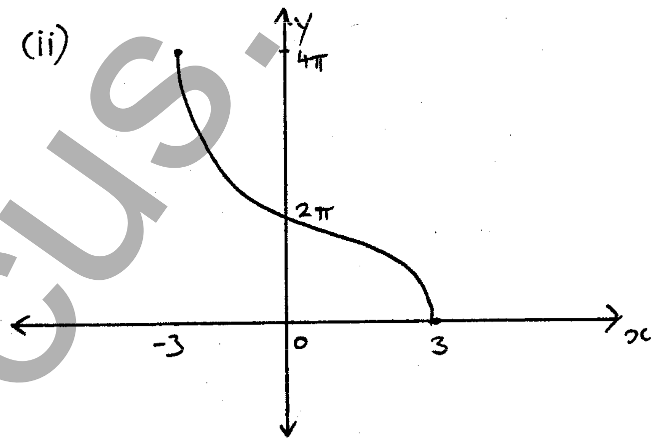
$\therefore D: -3 \le x \le 3$

Range is $0 \le \cos^{-1}\left(\frac{x}{3}\right) \le \pi$

$0 \le 4\cos^{-1}\left(\frac{x}{3}\right) \le 4\pi$

$0 \le y \le 4\pi.$

(ii)



(iii) $y = 4\cos^{-1}\left(\frac{x}{3}\right)$

$\frac{y}{4} = \cos^{-1}\left(\frac{x}{3}\right)$

$\cos\frac{y}{4} = \frac{x}{3} \quad \therefore x = 3\cos\frac{y}{4}$

$$\text{Area} = \int_0^{2\pi} 3\cos\frac{y}{4}\,dy$$

$$= 3 \times 4\left[\sin\frac{y}{4}\right]_0^{2\pi}$$

$$= 12\left[\sin\frac{2\pi}{4} - \sin 0\right]$$

$$= 12[1 - 0]$$

$$= 12 \text{ units}^2$$

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Question Two.

(a)

$$\frac{1}{|x-3|} \geq \frac{1}{2}$$

$$|x-3| \leq \frac{1}{2}$$

$$\sqrt{(x-3)^2} \leq \frac{1}{2}$$

$$(x-3)^2 \leq \frac{1}{4}$$

$$x^2-6x+9 \leq \frac{1}{4}$$

$$4x^2-24x+36 \leq 1$$

$$4x^2-24x+35 \leq 0$$

$$(2x-5)(2x-7) \leq 0$$

$$\therefore \quad \frac{5}{2} \leq x \leq \frac{7}{2}$$

(b) $O(0,0)$ $\quad P(2t, t^2)$

(i)

$$m_{OP} = \frac{t^2-0}{2t-0} = \frac{t}{2}$$

$$x^2 = 4y \quad \therefore y = \frac{1}{4}x^2$$

$$\therefore y' = \frac{1}{2}x$$

at $x = 2t$ $\quad y' = \frac{2t}{2} = t$

$$\therefore m_{PT} = t$$

(ii)

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$= \frac{t - \frac{t}{2}}{1 + t\left(\frac{t}{2}\right)}$$

$$= \frac{\frac{t}{2}}{1 + \frac{t^2}{2}} \times \left(\frac{2}{2}\right)$$

$$\tan\theta = \frac{t}{2+t^2}$$

(c)(i). circumference $= 2\pi r$

since $r = 1$ $\quad C = 2\pi$.

now, speed $= \dfrac{\text{distance}}{\text{time}}$

$$\therefore \text{speed} = \frac{2\pi}{1} = 2\pi$$

now, $\theta \propto$ arc length

$$\therefore \frac{d\theta}{dt} = 2\pi \text{ radians per sec.}$$

(ii) Area of $\Delta = \frac{1}{2}ab\sin C$.

$$A = \tfrac{1}{2}.r.r.\sin\theta$$

$$A = \tfrac{1}{2}r^2\sin\theta = \tfrac{1}{2}\sin\theta$$

now, $\dfrac{dA}{dt} = \dfrac{dA}{d\theta} \times \dfrac{d\theta}{dt}$

$$= \tfrac{1}{2}\cos\theta \times 2\pi.$$

at $\theta = \frac{2\pi}{3}$ $\quad \dfrac{dA}{dt} = \frac{1}{2}\cos\frac{2\pi}{3} \times 2\pi = \frac{1}{2}\left(-\frac{1}{2}\right) \times 2\pi$

$$= -\frac{\pi}{2}$$

$\therefore$ Area decreasing at $\frac{\pi}{2}$ cm²/sec.

Question Three.

(a) $u^2 = x.$

$\therefore u = x^{\frac{1}{2}} = \sqrt{x}$

$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$\frac{dx}{du} = 2\sqrt{x} \quad \therefore dx = 2\sqrt{x}\,du.$

$\therefore \int \frac{1}{\sqrt{x}(1+\sqrt{x})}\,dx = \int \frac{2\sqrt{x}}{\sqrt{x}(1+\sqrt{x})}\,du$

$= \int \frac{2}{1+u}\,du$

$= 2\log_e(1+u)$

$= 2\log_e(1+\sqrt{x}) + C.$

(b) $V_x = \pi \int_b^a y^2\,dx$

$= \pi \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \sin^2 x\,dx$

rearrange:
$\cos 2x = \cos^2 x - \sin^2 x$

$= \pi \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{1}{2} - \frac{1}{2}\cos 2x\,dx$

$= \pi \left[\frac{1}{2}x - \sin 2x\right]_{\frac{\pi}{12}}^{\frac{\pi}{4}}$

$= \pi \left[\frac{\pi}{8} - \sin\frac{\pi}{2} - \left(\frac{\pi}{24} - \sin\frac{\pi}{6}\right)\right]$

$= \pi \left[\frac{\pi}{8} - 1 - \frac{\pi}{24} + \frac{1}{2}\right]$

$= \pi \left[\frac{\pi}{12} + \frac{1}{2}\right]$

$= \frac{\pi^2}{12} + \frac{\pi}{2}$ units^3

(c)(i) $x = a\cos nt$

now $a = 10$ (particle starts 10m to right).

and $n = \pi$ (since $\frac{2\pi}{n} = 2$).

$\therefore x = 10\cos \pi t$

now 4 metres from starting point $x = 6$

since $v = -10\pi \sin \pi t$

and $\sin^2 \pi t + \cos^2 \pi t = 1$

$\therefore \left(\frac{v}{-10\pi}\right)^2 + \left(\frac{x}{10}\right)^2 = 1$

if $x = 6$ $\quad \frac{v^2}{100\pi^2} + \frac{36}{100} = 1$

$v^2 + 36\pi^2 = 100\pi^2$

$v^2 = 64\pi^2$

$v = \pm 8\pi$

$\therefore$ speed is 8π m/s

(ii) if $x = 6$, $\quad 6 = 10\cos \pi t$
(4 from start)

$\frac{6}{10} = \cos \pi t$

$\pi t = \cos^{-1}\frac{3}{5}$ (since only need 1st time)

$\therefore t = \frac{1}{\pi}\cos^{-1}\frac{3}{5}$

$t = 0.30$ seconds (2 dp)

Question Four

(a). $x^3 + ax^2 + 0x + 1 = 0$

(i) let roots be α, β, $\alpha+\beta$

$\therefore$ sum of roots $= -\frac{b}{a}$

$\alpha + \alpha + \beta + \beta = -\frac{a}{1}$

$\therefore 2(\alpha+\beta) = -a$

$\alpha + \beta = -\frac{a}{2}$

$\therefore$ one root is $-\frac{a}{2}$.

(ii) sum of roots (2 at a time) $= \frac{c}{a}$

$\therefore \alpha\beta + \alpha(\alpha+\beta) + \beta(\alpha+\beta) = \frac{0}{a}$

$\alpha\beta + \alpha^2 + \alpha\beta + \alpha\beta + \beta^2 = 0$

$3\alpha\beta + \alpha^2 + \beta^2 = 0$

$3\alpha\beta + (\alpha+\beta)^2 - 2\alpha\beta = 0$

$(\alpha+\beta)^2 + \alpha\beta = 0 \ldots$ ①

now $(\alpha)(\beta)(\alpha+\beta) = \frac{-d}{a}$

$\alpha\beta(\alpha+\beta) = -\frac{1}{1}$

$\alpha\beta\left(-\frac{a}{2}\right) = -1$

$\therefore \alpha\beta = -\frac{2}{a}$

$\therefore$ ① is $\left(-\frac{a}{2}\right)^2 - \frac{2}{a} = 0$

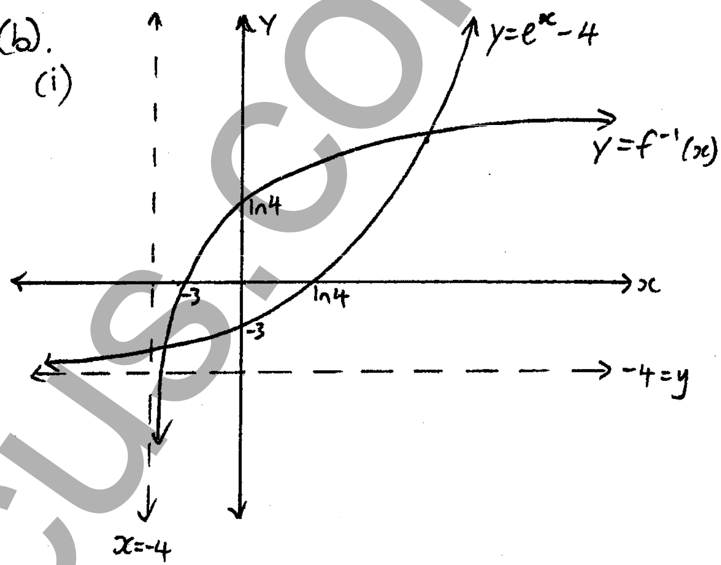
$\frac{a^2}{4} - \frac{2}{a} = 0$

$\frac{a^2}{4} = \frac{2}{a}$

$a^3 = 8$

$a = 2.$

(b).
(i)



(ii) See $y=f^{-1}(x)$ on diagram.

(iii) $y=f(x)$ passes through $y=x$.

$\therefore y=e^x-4$ solves with $y=x$

$\therefore x = e^x - 4 \ldots$ ①

Also, $y=f^{-1}(x)$ pass through $y=x$.

$\therefore y=f^{-1}(x)$ solves with $y=x$

② .... $x = f^{-1}(x)$ gives the x-value.

now, $e^x - x - 4 = 0$

(①=②) $\therefore x = e^x - 4$ which is the equation of $f(x) = f^{-1}(x)$ which solves for the x-value of the point of intersection.

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Question Four.

(iv) $e^{x} - x - 4 = 0$

at $x=0$ $\quad e^0 - 0 - 4 = 1 - 4 = -3$

at $x=2$ $\quad e^2 - 2 - 4 = e^2 - 6 = 1.39$

$\therefore$ root exists between $x=0$, $x=2$.

$\therefore \; x = \frac{0+2}{2} = 1$

at $x=1$ $\quad e^1 - 1 - 4 = e^1 - 5 = -2.28$

$\therefore$ root b/w $x=1$ and $x=2$

$\therefore \; x = \frac{1+2}{2} = 1.5$

at $x=1.5$ $\quad e^{1.5} - 1.5 - 4 = e^{1.5} - 5.5 = -1.02$

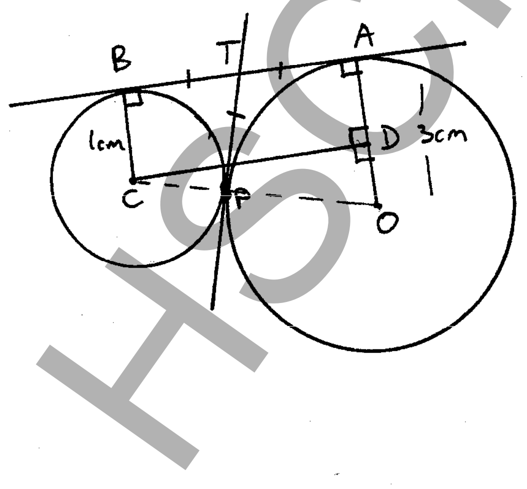
$\therefore$ root b/w $x=1.5$ and $x=2$

$\therefore$ root is $x=2$ (to n. integer).

Question Five.

(a)

(i)



(ii) construct CO.

now $CO = 4cm$ (sum of both radii).

$DO = 3 - 1 = 2cm$ (since $AD = BC$)

$\therefore \; CD^2 + DO^2 = CO^2$ (Pythagoras).

$CD^2 + 2^2 = 4^2$

$CD^2 = 12 \quad \therefore \; CD = 2\sqrt{3}\,cm$

$\therefore \; \underline{AB = 2\sqrt{3}\,cm}$ (since $AB = CD$).

(iii) $BT = TP = AT$ (tangents from an ext. point are equal).

$\therefore \; 2PT = 2TA$

$\therefore \; 2PT = AB$ (since $TA = TB$).

$\therefore \; 2PT = 2\sqrt{3}$ (from ii).

$\underline{PT = \sqrt{3}\,cm.}$

(b) (i) $v = (1-x)^2$

$\dot{v} = a = 2(1-x)^1 \times -1 = \underline{-2(1-x)}$

(ii) $v = (1-x)^2$

$\therefore \; \frac{dx}{dt} = (1-x)^2$

$\frac{dt}{dx} = (1-x)^{-2}$

$\int \frac{dt}{dx}\,dx = \int (1-x)^{-2}\,dx$

$t = \frac{(1-x)^{-1}}{-1 \times -1} + c$

$t = \frac{1}{1-x} + c$

at $t=0$ $x=0$ $\quad \therefore \; 0 = \frac{1}{1+0} + c \quad \therefore \; c = -1$

PTO

Question Five.

(ii)

$$\therefore \quad t = \frac{1}{1-x} - 1$$

$$t+1 = \frac{1}{1-x}$$

$$1-x = \frac{1}{t+1}$$

$$\therefore x = 1 - \frac{1}{t+1}$$

(iii) $x = 1 - (t+1)^{-1}$

$$v = \frac{1}{(t+1)^2}$$

when $t=0$ $v = 1$

$\therefore$ 1% of initial speed $= 1\%$ of 1 m/sec.

$= \frac{1}{100}$ m/sec

$\therefore$ let $v = \frac{1}{100}$ $t = ?$

$$\frac{1}{100} = \frac{1}{(t+1)^2}$$

$$\therefore \pm 10 = t+1$$

$t = -1 \pm 10$ $t = -11$ or 9

$\therefore t = 9$ seconds

Question Six.

(a)(i)

$$(1+x)^n = \sum_{r=0}^{n} {}^nC_r (1)^{n-r} (x)^r$$

$$= \sum_{r=0}^{n} {}^nC_r x^r$$

$$= 1 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \dots + {}^nC_n x^n$$

(ii)

$${}^nC_4 = 2\ {}^nC_3$$

$$\frac{n!}{(n-4)!\,4!} = \frac{2n!}{(n-3)!\,3!}$$

$$\frac{\cancel{n!}}{\cancel{(n-4)!}\,4\times\cancel{3!}} = \frac{2\cancel{n!}}{(n-3)\cancel{(n-4)!}\,\cancel{3!}}$$

$$\frac{1}{4} = \frac{2}{n-3}$$

$n-3 = 8$ $\therefore n = 5$

(b)(i)

MAA $= 2\times4\times3 = 24$

AAM $= (5\times3\times2)2 = 60$

AMA $= (5\times1\times3)2 = 30$ $\therefore$ 174 ways.

or $= (5\times2\times3)2 = 60$

(ii) $(2\times3\times2)3! = 72$ ways

(c) (i) P(at most one colourblind)

$=$ P(none) or P(one)

$$= {}^{20}C_0 \left(\tfrac{5}{100}\right)^0 \left(\tfrac{95}{100}\right)^{20} + {}^{20}C_1 \left(\tfrac{5}{100}\right)^1 \left(\tfrac{95}{100}\right)^{19}$$

$= 0.74$

(ii) P(at least 2 colourblind)

$= 1 -$ P(at most one colourblind)

$= 1 - 0.74$

$= 0.26$

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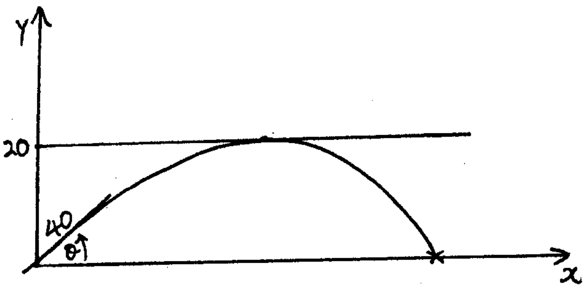
Question Seven.

(a).



$x = 40t\cos\theta \qquad y = 40t\sin\theta - 5t^2$

max height when $\dot{y} = 0$

$\therefore 40\sin\theta - 10t = 0$

$\therefore t = 4\sin\theta.$

$\therefore$ when $t = 4\sin\theta \quad y = 20.$

$\therefore \quad 20 = 40\sin\theta(4\sin\theta) - 5(4\sin\theta)^2$

$20 = 160\sin^2\theta - 80\sin^2\theta$

$20 = 80\sin^2\theta$

$\therefore \sin\theta = \pm\frac{1}{2}$

$\therefore \theta = \frac{\pi}{6} \qquad (\text{since } \theta < 90°).$

let $y=0 \quad \therefore 40t\sin\theta - 5t^2 = 0 \quad \therefore t = 8\sin\theta$

but $\theta = \frac{\pi}{6}$

$\therefore$ max horizontal range when $t=4.$

$\therefore \quad x = 40t\cos\theta$

$x = 40(4)\cos\frac{\pi}{6}$

$= 160\left(\frac{\sqrt{3}}{2}\right)$

$= \underline{80\sqrt{3}\text{ m.}}$

(b)(i) Prove $(1+x)^n - 1$ is divisible by x.

Step 1. Prove true for $n=1$.

$LHS = (1+x)^1 - 1$

$= x$

which is divisible by x.

Step 2. Assume true for $n=k$.

ie, $(1+x)^k - 1 = x.Q(x)$

$\therefore (1+x)^k = 1 + x.Q(x)$

To prove true for $n=k+1$.

ie $(1+x)^{k+1} - 1 = x.P(x)$

$LHS = (1+x)^k(1+x) - 1$

$= [1 + x.Q(x)](1+x) - 1$

$= (1+x) + (1+x).xQ(x) - 1$

$= x + (1+x).x.Q(x)$

$= x[1 + (1+x).Q(x)]$

$= x.P(x) \quad$ where $P(x) = 1 + (1+x).Q(x)$

which is divible by x.

Step 3. OH BABY, BY PMI.

b(ii) $12^n - 4^n - 3^n + 1$

$= 4^n.3^n - 4^n - 3^n + 1$

$= 4^n(3^n - 1) - (3^n - 1)$

$= (3^n - 1)(4^n - 1)$

$= \underbrace{[(2+1)^n - 1]}_{\div \text{ by } 2}\underbrace{[(3+1)^n - 1]}_{\div \text{ by } 3}$ using (b i) above.

$= [(3^n-1)][4^n-1]$ must be divisible by 6.

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